The Lifted Root Number Conjecture for small sets of places

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Abstract

Let L/K be a finite Galois extension of number fields with Galois group G. The Lifted Root Number Conjecture (LRNC) by K.W. Gruenberg, J. Ritter and A. Weiss relates the leading terms at zero of Artin L-functions attached to L/K to natural arithmetic invariants. D. Burns used complexes arising from étale cohomology of the constant sheaf $\mathbb{Z}$ to define a canonical element $T\Omega(L/K)$ of the relative K-group $K_0(\mathbb{Z}G, \mathbb{R})$. It was shown that the LRNC for L/K is equivalent to the vanishing of $T\Omega(L/K)$ and that this in turn is equivalent to the Equivariant Tamagawa Number Conjecture for the pair $(h^0(\mathrm{Spec}(L))(0), \mathbb{Z}G)$. These conjectures make use of a finite G-invariant set S of places of L which is supposed to be sufficiently large. We formulate a LRNC for small sets S which only need to contain the archimedean primes and give an application to a special class of CM-extensions.

Let L/K be a finite Galois extension of number fields with Galois group G and S a finite G-invariant set of places of L which contains the set S_∞ of all the archimedean primes. In [RW96] the authors derive an exact sequence of finitely generated $\mathbb{Z}G$-modules

$$E_S \rightarrowtail A \to B \twoheadrightarrow \nabla, \tag{1}$$

which has a uniquely determined extension class in $\mathrm{Ext}_G^2(\nabla, E_S)$. Note that the sequence itself is not unique. We will refer to a sequence (1) as a Tate-sequence for S. Here, E_S is the group of S-units of L, A is c.t. (short for cohomologically trivial), B projective and ∇ fits into an exact sequence of G-modules

$$\mathrm{cl}_S \rightarrowtail \nabla \twoheadrightarrow \overline{\nabla}.$$

Indeed, the S-class group of L is the torsion submodule of ∇, hence $\overline{\nabla}$ is a $\mathbb{Z}G$-lattice. If S is large in the sense that all ramified primes lie in S and $\mathrm{cl}_S = 1$, the modules ∇ and $\overline{\nabla}$ coincide and are just the kernel ΔS of the augmentation map $\mathbb{Z}S \twoheadrightarrow \mathbb{Z}$. In this case, the extension class of (1) is Tate's canonical class ([Ta66]).

Starting with an equivariant injection $\phi : \Delta S \rightarrowtail E_S$ for large S, an arithmetic invariant $\Omega_\phi \in K_0T(\mathbb{Z}G)$ is defined in [GRW99]; Ω_ϕ essentially is the class of the cokernel of an injection $\tilde{\phi} : B \rightarrowtail A$ constructed via ϕ. Assuming the validity of Stark's conjecture the LRNC states that Ω_ϕ is determined by a homomorphism

$$\chi \mapsto W(L/K, \check{\chi}) R_\phi(\check{\chi}) / c_S(\check{\chi})$$

on the ring of virtual characters of G. Here, $W(L/K, \chi)$ is defined in terms of Artin root numbers, R_ϕ is the Stark-Tate regulator and $c_S(\chi)$ is the leading coefficient of the Taylor expansion of the S-truncated L-function $L_S(L/K, \chi, s)$ at $s = 0$. D. Burns [Bu01] proved that the LRNC is equivalent to the Equivariant Tamagawa Number Conjecture for the pair $(h^0(\mathrm{Spec}(L))(0), \mathbb{Z}G)$ which is known

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to be true if L is absolutely abelian (cf. [BG03, Fl02]).

If S is small, one cannot copy the construction of Ω_ϕ, since in general there do not exist injections $\nabla \rightarrowtail E_S$. But there always exist equivariant isomorphisms $\phi : \mathbb{Q}\overline{\nabla} \to \mathbb{Q}(E_S \oplus C)$ with an appropriate free $\mathbb{Z}G$-module C. We transpose ϕ to an isomorphism $\tilde{\phi} : \mathbb{Q}B \to \mathbb{Q}(A \oplus C)$ and (essentially) define Ω_ϕ to be $(B, \tilde{\phi}, A \oplus C) \in K_0(\mathbb{Z}G, \mathbb{Q}) \simeq K_0T(\mathbb{Z}G)$. After this is done in section 2, we discuss variance with ϕ and S in section 3. We define a modified version of the Stark-Tate regulator and state a LRNC for small S in section 4. Finally, we give an application to "nice" CM-extensions which were introduced by C. Greither [Gr00]. We point out that this paper includes parts of the author's dissertation [Ni08].

1. Preliminaries

1.0.1 *Duals* Let G be a finite group. For each left[1] $\mathbb{Z}G$-module M we write M^0 for its $\mathbb{Z}$-dual $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ with the G-action formula $(gf)(m) = gf(g^{-1}m) = f(g^{-1}m)$ for $g \in G$, $f \in M^0$ and $m \in M$. Note that there is a natural $\mathbb{Z}G$-isomorphism $\mathbb{Z}G \simeq \mathbb{Z}G^0$ that sends each $g \in G$ to the homomorphism $h \mapsto \delta_{gh}$. Of course, the δ on the righthand side is Kronecker's. Under this identification, the dual of the natural augmentation map $\mathbb{Z}G \twoheadrightarrow \mathbb{Z}$ is the map $\mathbb{Z} \rightarrowtail \mathbb{Z}G$ that sends 1 to $N_G = \sum_{g \in G} g$. Thus, we get a $\mathbb{Z}G$-isomorphism

$$\Delta G^0 \simeq \mathbb{Z}G/N_G, \tag{2}$$

where ΔG denotes the kernel of the augmentation map.

1.0.2 *K-theory* Let R be a left noetherian ring with 1 and $\mathrm{PMod}(R)$ the category of all finitely generated projective R-modules. We write $K_0(R)$ for the Grothendieck group of $\mathrm{PMod}(R)$, and $K_1(R)$ for the Whitehead group of R which is the abelianized infinite general linear group. If S is a multiplicatively closed subset of the center of R which contains no zero divisors, $1 \in S$, $0 \notin S$, we denote the Grothendieck group of the category of all finitely generated S-torsion R-modules of finite projective dimension by $K_0S(R)$. Writing R_S for the ring of quotients of R with denominators in S we have the Localization Sequence (cf. [CR87], p. 65)

$$K_1(R) \to K_1(R_S) \xrightarrow{\partial} K_0S(R) \to K_0(R) \to K_0(R_S). \tag{3}$$

If T is a ring that contains R and M is an R-module, we will often write TM instead of $T \otimes_R M$. Moreover, if G is a group and $M = \Delta G$ is the kernel of the augmentation map $RG \twoheadrightarrow R$, we set $\Delta_T G := T \otimes_R \Delta G$.
Specializing to group rings $\mathbb{Z}G$ for finite groups G and $S = \mathbb{Z} \setminus \{0\}$ we write $K_0T(\mathbb{Z}G)$ instead of $K_0S(\mathbb{Z}G)$. So (3) reads

$$K_1(\mathbb{Z}G) \to K_1(\mathbb{Q}G) \xrightarrow{\partial} K_0T(\mathbb{Z}G) \to K_0(\mathbb{Z}G) \to K_0(\mathbb{Q}G). \tag{4}$$

Note that a finitely generated $\mathbb{Z}G$-module has finite projective dimension if and only if it is a G-c.t. module. Indeed, the projective dimension is less or equal to 1 in this case. Further, recall that the relative K-group $K_0(\mathbb{Z}G, \mathbb{Q})$ is generated by elements of the form (P_1, ϕ, P_2) with finitely generated projective $\mathbb{Z}G$- modules P_1 and P_2 and a $\mathbb{Q}G$-isomorphism $\phi : \mathbb{Q}P_1 \to \mathbb{Q}P_2$, and that there is an isomorphism (cf. [Sw68])

$$i_G : K_0T(\mathbb{Z}G) \simeq K_0(\mathbb{Z}G, \mathbb{Q}). \tag{5}$$

[1] all occurring modules in this paper are left modules

If a c.t. torsion $\mathbb{Z}G$-module T has projective resolution $P_1 \overset{\iota}{\rightarrowtail} P_0 \twoheadrightarrow T$, this isomorphism sends the corresponding element $[T] \in K_0T(\mathbb{Z}G)$ to $(P_1, \mathbb{Q}\otimes\iota, P_0) \in K_0(\mathbb{Z}G,\mathbb{Q})$.
If p is a finite rational prime, the local analogue of sequence (4) is

$$K_1(\mathbb{Z}_pG) \to K_1(\mathbb{Q}_pG) \xrightarrow{\partial_p} K_0T(\mathbb{Z}_pG) \to 0, \tag{6}$$

and we have an isomorphism

$$K_0T(\mathbb{Z}G) \simeq \bigoplus_{p\nmid\infty} K_0T(\mathbb{Z}_pG). \tag{7}$$

1.0.3 *Complexes and refined Euler Characteristics* For any ring R we write $\mathcal{D}(R)$ for the derived category of R-modules. Let $\mathcal{C}^b(\mathrm{PMod}(R))$ be the category of bounded complexes of finitely generated projective R-modules. A complex of R-modules is called perfect if it is isomorphic in $\mathcal{D}(R)$ to an element of $\mathcal{C}^b(\mathrm{PMod}(R))$. We denote the full triangulated subcategory of $\mathcal{D}(R)$ consisting of perfect complexes by $\mathcal{D}^{\mathrm{perf}}(R)$. For any $C^\cdot \in \mathcal{D}^{\mathrm{perf}}(R)$ we define R-modules

$$C^e := \bigoplus_{i\in\mathbb{Z}} C^{2i},\ C^o := \bigoplus_{i\in\mathbb{Z}} C^{2i+1}.$$

For the following let R be a Dedekind domain of characteristic 0, K its field of fractions, A a finite dimensional K-algebra and Γ an R-order in A. A pair $(C^\cdot, t)$ consisting of a complex $C^\cdot \in \mathcal{D}^{\mathrm{perf}}(\Gamma)$ and an isomorphism $t : H^o(C^\cdot_K) \to H^e(C^\cdot_K)$ is called a trivialised complex, where $C^\cdot_K$ is the complex obtained by tensoring $C^\cdot$ with K. We refer to t as a trivialisation of $C^\cdot$.
One defines the refined Euler characteristic $\chi_{\Gamma,A}(C^\cdot, t) \in K_0(\Gamma, A)$ of a trivialised complex as follows: Choose a complex $P^\cdot \in \mathcal{C}^b(\mathrm{PMod}(R))$ which is quasi-isomorphic to $C^\cdot$. Let $B^i(P^\cdot_K)$ and $Z^i(P^\cdot_K)$ denote the i^{th} cobounderies and i^{th} cocycles of $P^\cdot_K$, respectively. We have the obvious exact sequences

$$B^i(P^\cdot_K) \rightarrowtail Z^i(P^\cdot_K) \twoheadrightarrow H^i(P^\cdot_K)\ ,\quad Z^i(P^\cdot_K) \rightarrowtail P^i_K \twoheadrightarrow B^{i+1}(P^\cdot_K).$$

If we choose splittings of the above sequences we get an isomorphism

$$\begin{aligned} \phi_t : P^o_K &\simeq \textstyle\bigoplus_{i\in\mathbb{Z}} B^i(P^\cdot_K) \oplus H^o(P^\cdot_K) \\ &\simeq \textstyle\bigoplus_{i\in\mathbb{Z}} B^i(P^\cdot_K) \oplus H^e(P^\cdot_K) \\ &\simeq P^e_K, \end{aligned}$$

where the second map is induced by t. Then the refined Euler characteristic is defined to be

$$\chi_{\Gamma,A}(C^\cdot, t) := (P^o, \phi_t, P^e) \in K_0(\Gamma, A)$$

which indeed is independent of all choices made in the construction.
Now we specialize to group rings RG, where R is a finitely generated subring of $\mathbb{Q}$. Let H^i, $i = 0, 1$ be finitely generated RG-modules and

$$H^0 \rightarrowtail A \to B \twoheadrightarrow H^1$$

an exact sequence representing an extension class $\tau \in \mathrm{Ext}^2_{RG}(H^1, H^0)$. One obtains an associated complex $A \to B$, where A is placed in degree 0. If this complex is perfect, τ is called a perfect 2-extension. Moreover, if there is a $\mathbb{Q}G$-isomorphism $\phi : \mathbb{Q}H^1 \to \mathbb{Q}H^0$, the element

$$\chi_{RG,\mathbb{Q}G}(\tau, \phi) := \chi_{RG,\mathbb{Q}G}(A \to B, \phi)$$

only depends upon the class τ and the isomorphism ϕ. For further information concerning refined Euler characteristics we refer the reader to [Bu03].

Definition 1.1. Let A be a finitely generated c.t. $\mathbb{Z}G$-module, B projective and $\phi : \mathbb{Q}A \to \mathbb{Q}B$ a $\mathbb{Q}G$-isomorphism.We define:

$$(A, \phi, B) = -(B, \phi^{-1}, A) := \chi_{\mathbb{Z}G,\mathbb{Q}G}(C^\cdot, \phi) \in K_0(\mathbb{Z}G, \mathbb{Q}),$$

where $C^\cdot$ is the perfect complex $\ldots \to 0 \to A \to B \to 0 \to \ldots$, and the position of A is in degree -1 and all maps are zero.

Note that this coincides with the usual definition.

Remark 1. i) If A is a c.t. torsion $\mathbb{Z}G$-module, then $i_G([A]) = -(A, 0, 0) = (0, 0, A) \in K_0(\mathbb{Z}G, \mathbb{Q})$.

ii) We can replace $K_0(\mathbb{Z}G, \mathbb{Q})$ by $K_0(\mathbb{Z}_pG, \mathbb{Q}_p)$ for any prime p. Everything remains the same except for the obvious modifications.

1.0.4 Hom *description* Let G be a finite group, p a finite rational prime and $R(G)$ (resp. $R_p(G)$) the ring of virtual characters of G with values in $\mathbb{Q}^c$ (resp. $\mathbb{Q}_p^c$), an algebraic closure of $\mathbb{Q}$ (resp. $\mathbb{Q}_p$). Choose a number field F, Galois over $\mathbb{Q}$ with Galois group Γ, which is large enough such that all representations of G can be realized over F. Let $\wp$ be a prime of F above p. Then there is an isomorphism (for this and the following cf. [GRW99], Appendix A)

$$\begin{array}{rcl} \mathrm{Det} : K_1(\mathbb{Q}_pG) & \stackrel{\simeq}{\longrightarrow} & \mathrm{Hom}_{\Gamma_\wp}(R_p(G), F_\wp^\times) \\ {[X, g]} & \mapsto & [\chi \mapsto \det(g|\mathrm{Hom}_{F_\wp G}(V_\chi, F_\wp \otimes_{\mathbb{Q}_p} X))], \end{array}$$

where V_χ is a $F_\wp G$-module with character χ. Combined with the localization sequence (6) this gives the local Hom description

$$K_0T(\mathbb{Z}_pG) \simeq \mathrm{Hom}_{\Gamma_\wp}(R_p(G), F_\wp^\times)/\mathrm{Det}\,(\mathbb{Z}_pG^\times). \tag{8}$$

One globally has

$$K_0T(\mathbb{Z}G) \simeq \mathrm{Hom}_\Gamma^+(R(G), J_F)/\mathrm{Det}\,U(\mathbb{Z}G), \tag{9}$$

where J_F denotes the idèle group of F and $U(\mathbb{Z}G)$ the unit idèles of $\mathbb{Z}G$. The $+$ indicates that a homomorphism in $\mathrm{Hom}_\Gamma^+(R(G), J_F)$ takes values in $\mathbb{R}^+$ for symplectic characters.

2. Outline of the construction

Let L/K be a Galois extension of number fields with Galois group G. For a prime $\mathfrak{P}$ of L we write $\mathfrak{p} = \mathfrak{P} \cap K$ for the prime below $\mathfrak{P}$, $G_\mathfrak{P}$ for the decomposition group attached to $\mathfrak{P}$ and $I_\mathfrak{P}$ for the inertia subgroup. We denote the Frobenius generator of the Galois group $\overline{G_\mathfrak{P}} = G_\mathfrak{P}/I_\mathfrak{P}$ of the corresponding residue field extension by $\phi_\mathfrak{P}$.

The inertial lattice of the local extension $L_\mathfrak{P}/K_\mathfrak{p}$ is defined to be the $\mathbb{Z}G_\mathfrak{P}$-lattice (cf. [GW96] or [We96] p. 42)

$$W_\mathfrak{P} = \{(x, y) \in \Delta G_\mathfrak{P} \oplus \mathbb{Z}\overline{G_\mathfrak{P}} : \overline{x} = (\phi_\mathfrak{P} - 1)y\}. \tag{10}$$

Note that $W_\mathfrak{P} \simeq \mathbb{Z}G_\mathfrak{P}$ if the local extension $L_\mathfrak{P}/K_\mathfrak{p}$ is unramified. Projecting on the first component yields an exact sequence of $G_\mathfrak{P}$-modules

$$\mathbb{Z} \rightarrowtail W_\mathfrak{P} \twoheadrightarrow \Delta G_\mathfrak{P}. \tag{11}$$

The $\mathbb{Z}$-dual of this sequence induces a surjection $W_\mathfrak{P}^0 \twoheadrightarrow \mathbb{Z}^0 = \mathbb{Z}$. If we combine these surjections and the augmentation map $\mathbb{Z}S \twoheadrightarrow \mathbb{Z}$, we get an exact sequence

$$\overline{\nabla} \rightarrowtail \mathbb{Z}S \oplus \bigoplus_{\mathfrak{P} \in S_{\mathrm{ram}}^* \setminus (S \cap S_{\mathrm{ram}})^*} \mathrm{ind}_{G_\mathfrak{P}}^G(W_\mathfrak{P}^0) \twoheadrightarrow \mathbb{Z} \tag{12}$$

where the $*$ indicates that the sum runs over a fixed set of representatives, one for each orbit of the action of G on the primes of L. Due to this characterization of $\overline{\nabla}$ we have

Lemma 2.1. *Let L/K be a finite Galois extension of number fields with Galois group G and S a finite G-invariant set of places of L which contains all the archimedean primes. Moreover, let C be a free $\mathbb{Z}G$-module of rank $|S^*_{\mathrm{ram}} \backslash (S \cap S_{\mathrm{ram}})^*|$. Then there exist $\mathbb{Q}G$-isomorphisms $\mathbb{Q}\overline{\nabla} \xrightarrow{\simeq} \mathbb{Q}(E_S \oplus C)$.*

Proof. □

In order to get an element $\Omega_\phi \in K_0(\mathbb{Z}G, \mathbb{Q})$ analogously to the Ω_ϕ of [GRW99], we split sequence (1) into two parts:

$$E_S \rightarrowtail A \twoheadrightarrow W \text{ and } W \rightarrowtail B \twoheadrightarrow \nabla \tag{13}$$

We will refer to it as the left and the right part of the Tate-sequence. From the construction of the Tate-sequence for small sets S one gets the following diagram, which we can take for a definition of the $\mathbb{Z}G$-lattice R:

$$\begin{array}{ccccc} W & \hookrightarrow & B & \twoheadrightarrow & \nabla \\ \downarrow{\scriptstyle i} & & \| & & \downarrow \\ R & \hookrightarrow & B & \twoheadrightarrow & \overline{\nabla} \\ \downarrow & & & & \\ \mathrm{cl}_S & & & & \end{array} \tag{14}$$

We now choose $\mathbb{Q}G$-automorphisms α of $\mathbb{Q}W$ and β of $\mathbb{Q}R$ as well as $\mathbb{Q}G$-isomorphisms $\tilde{\alpha}$ and $\tilde{\beta}$ making the following diagrams commutative:

$$\begin{array}{ccccc} \mathbb{Q}(E_S \oplus C) & \hookrightarrow & \mathbb{Q}(E_S \oplus C \oplus W) & \twoheadrightarrow & \mathbb{Q}W \\ \| & & \downarrow{\scriptstyle \tilde{\alpha}} & & \downarrow{\scriptstyle \alpha} \\ \mathbb{Q}(E_S \oplus C) & \hookrightarrow & \mathbb{Q}(A \oplus C) & \twoheadrightarrow & \mathbb{Q}W \end{array} \tag{15}$$

$$\begin{array}{ccccc} \mathbb{Q}R & \hookrightarrow & \mathbb{Q}B & \twoheadrightarrow & \mathbb{Q}\overline{\nabla} \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \tilde{\beta}} & & \| \\ \mathbb{Q}R & \hookrightarrow & \mathbb{Q}(R \oplus \overline{\nabla}) & \twoheadrightarrow & \mathbb{Q}\overline{\nabla} \end{array} \tag{16}$$

In diagram (15) C is a free $\mathbb{Z}G$-module as in Lemma 2.1. The lower sequence derives from adding C to the left part of the Tate-sequence. The upper sequence is the canonical one as well as the lower sequence in (16). The upper sequence in (16) is extracted from (14).
Given a $\mathbb{Q}G$-isomorphism $\phi : \mathbb{Q}\overline{\nabla} \to \mathbb{Q}(E_S \oplus C)$ as in Lemma 2.1 we define a $\mathbb{Q}G$-isomorphism $\tilde{\phi}$ to be the composite map

$$\tilde{\phi} : \mathbb{Q}B \xrightarrow{\tilde{\beta}} \mathbb{Q}(R \oplus \overline{\nabla}) \xrightarrow{\mathrm{id}_R \oplus \phi} \mathbb{Q}(R \oplus E_S \oplus C) \tag{17}$$

$$\xrightarrow{i^{-1} \oplus \mathrm{id}_{E_S \oplus C}} \mathbb{Q}(W \oplus E_S \oplus C) \xrightarrow{\tilde{\alpha}} \mathbb{Q}(A \oplus C).$$

We define

$$\boxed{\Omega_\phi := (B, \tilde{\phi}, A \oplus C) - \partial[\mathbb{Q}W, \alpha] - \partial[\mathbb{Q}R, \beta] \in K_0(\mathbb{Z}G, \mathbb{Q}).} \tag{18}$$

Remark 2. i) One can choose the isomorphisms α and β to be the identity on $\mathbb{Q}W$ and $\mathbb{Q}R$, respectively. Sometimes, however, it may be useful to choose injections $W \rightarrowtail W$ and $R \rightarrowtail R$, homotopic to 0, since we can actually build $\mathbb{Z}G$-diagrams corresponding to those in (15) and (16) in this case. These injections automatically become isomorphisms after tensoring with $\mathbb{Q}$. If S is large, this also shows that our construction yields the Ω_ϕ of [GRW99].

ii) The natural homomorphism $K_0(\mathbb{Z}G, \mathbb{Q}) \to K_0(\mathbb{Z}G)$ induced by i_G and the Localization sequence (4) sends Ω_ϕ to Chinburg's $\Omega_3(L/K)$ (cf. [Ch85], p. 357 or [We96]).

We have defined an element Ω_ϕ attached to the following data (D):

– a finite Galois extension L/K of number fields with Galois group G,
– a finite G-invariant set S of places of L which contains all the infinite primes,
– a $\mathbb{Q}G$-isomorphism $\phi : \mathbb{Q}\overline{\nabla} \to \mathbb{Q}(E_S \oplus C)$, where $\overline{\nabla}$ is the leftmost term in sequence (12) and C is a free $\mathbb{Z}G$-module of rank $|S_{\mathrm{ram}}^* \setminus (S \cap S_{\mathrm{ram}})^*|$ as in Lemma 2.1.

Theorem 2.2. *The data (D) uniquely determine an element $\Omega_\phi \in K_0(\mathbb{Z}G, \mathbb{Q})$.*

We divide the proof into two lemmas.

Lemma 2.3. *The definition of Ω_ϕ is independent of the choices of α, β, $\tilde{\alpha}$ and $\tilde{\beta}$.*

Proof. □

Secondly, we have to check:

Lemma 2.4. *The definition of Ω_ϕ is independent of the choice of the Tate-sequence.*

Proof. □

3. Basic properties of Ω_ϕ

In this section we discuss variance of the isomorphism ϕ and of the set of places S. The most interesting (and most complicated) case is, how Ω_ϕ varies if S is enlarged by ramified primes. The following proposition describes variance with ϕ and is the analogue of Proposition 1 in [GRW99].

Proposition 3.1. *Fix a set of data (D), and let $\phi' : \mathbb{Q}\overline{\nabla} \to \mathbb{Q}(E_S \oplus C)$ be another $\mathbb{Q}G$-isomorphism. Then*

$$\Omega_{\phi'} - \Omega_\phi = \partial[\mathbb{Q}\overline{\nabla}, \phi^{-1} \circ \phi'].$$

In particular, $\Omega_{\phi'} - \Omega_\phi$ has representing homomorphism

$$\chi \mapsto \det(\phi^{-1} \circ \phi' | \mathrm{Hom}_{\mathbb{C}G}(V_\chi, \mathbb{C}\overline{\nabla})),$$

where V_χ is a $\mathbb{C}G$-module with character χ.

Proof. □

Our next task is to enlarge S by a ramified prime $\mathfrak{P}_0$, i.e. $\mathfrak{P}_0 \in S_{\mathrm{ram}}$, but $\mathfrak{P}_0 \notin S$. We may assume $\mathfrak{P}_0 \in S_{\mathrm{ram}}^*$.

Note that some of the ideas in what follows are taken from [Gr07], where the author assumes the validity of the LRNC for an abelian CM-extension L/K to compute the Fitting ideal of $(\mathrm{cl}_L^-)^\vee$, the Pontryagin dual of the minus class group of L. For this, he connects a Tate-sequence for a large set S of places of L to a Tate-sequence for S_∞. In what follows here, some of the maps between Tate-sequences are inspired by the corresponding maps in [Gr07]. But some of the diagrams in loc. cit. only commute on minus parts owing to the purpose of this paper; so we have to modify the construction in order to achieve commutative diagrams in general. Moreover, the author does not introduce an element like Ω_ϕ, nor does he give a definition of a modified Stark-Tate regulator, as we intend to do in the next section.

We set $S_0 := S \cup G\mathfrak{P}_0$ and we intend to indicate each module by a subscript S resp. S_0 (or simply a subscript 0) if it is not clear to which (construction of a) Tate-sequence it belongs.
The dual of sequence (11) for the prime $\mathfrak{P}_0$ yields the following commutative diagram:

$$\begin{array}{ccccc}
\operatorname{ind}_{G_{\mathfrak{P}_0}}^G \Delta G_{\mathfrak{P}_0}^0 & = & \operatorname{ind}_{G_{\mathfrak{P}_0}}^G \Delta G_{\mathfrak{P}_0}^0 & & \\
\downarrow & & \downarrow & & \\
\overline{\nabla}_S & \hookrightarrow & \mathbb{Z}S \oplus \bigoplus_{\mathfrak{P}\in S_{\mathrm{ram}}^*\setminus(S\cap S_{\mathrm{ram}})^*} \operatorname{ind}_{G_\mathfrak{P}}^G W_\mathfrak{P}^0 & \twoheadrightarrow & \mathbb{Z} \\
\downarrow & & \downarrow & & \| \\
\overline{\nabla}_{S_0} & \hookrightarrow & \mathbb{Z}S_0 \oplus \bigoplus_{\mathfrak{P}\in S_{\mathrm{ram}}^*\setminus(S_0\cap S_{\mathrm{ram}})^*} \operatorname{ind}_{G_\mathfrak{P}}^G W_\mathfrak{P}^0 & \twoheadrightarrow & \mathbb{Z}
\end{array}$$

We extract the left column and use (2) to get an exact sequence

$$\mathbb{Z}G/N_{G_{\mathfrak{P}_0}} \hookrightarrow \overline{\nabla}_S \xrightarrow{\pi_{\overline{\nabla}}} \overline{\nabla}_{S_0}. \tag{19}$$

Let $h_L = |\mathrm{cl}_L|$ be the class number of L and choose a positive integer h such that $h_L | h$. Then $\mathfrak{P}_0^h$ is principal generated by a S_0-unit $u_{\mathfrak{P}_0}$. Let us define a map (which is the map β in [Gr07])

$$u_0 : \mathbb{Z}G \to E_{S_0},\ 1 \mapsto u_{\mathfrak{P}_0}.$$

Then we have a left exact sequence

$$\mathbb{Z}G \cdot \Delta G_{\mathfrak{P}_0} \xhookrightarrow{(-u_0,\mathrm{id})} E_S \oplus \mathbb{Z}G \xrightarrow{(\mathrm{id},u_0)} E_{S_0}, \tag{20}$$

since for $x \in \mathbb{Z}G$ we have $x \cdot u_{\mathfrak{P}_0} \in E_S$ if and only if $x \in \mathbb{Z}G \cdot \Delta G_{\mathfrak{P}_0}$. Moreover, we have a $\mathbb{Q}G$-isomorphism

$$\begin{array}{rccl}
\phi' : & \mathbb{Q}G/N_{G_{\mathfrak{P}_0}} & \to & \mathbb{Q}G \cdot \Delta G_{\mathfrak{P}_0}, \\
& 1 \bmod N_{G_{\mathfrak{P}_0}} & \mapsto & 1 - \frac{1}{|G_{\mathfrak{P}_0}|} N_{G_{\mathfrak{P}_0}}.
\end{array} \tag{21}$$

Let C_0 be a free $\mathbb{Z}G$-module of rank $|S_{\mathrm{ram}}^* \setminus (S_0 \cap S_{\mathrm{ram}})^*|$, and start with a $\mathbb{Q}G$-isomorphism $\phi_0 : \mathbb{Q}\overline{\nabla}_{S_0} \to \mathbb{Q}(E_{S_0} \oplus C_0)$. Then one can always find a $\mathbb{Q}G$-isomorphism ϕ fitting in a commutative diagram

$$\begin{array}{ccc}
\mathbb{Q}G/N_{G_{\mathfrak{P}_0}} & \xrightarrow{\phi'} & \mathbb{Q}G \cdot \Delta G_{\mathfrak{P}_0} \\
\downarrow & & \downarrow {\scriptstyle (-u_0,\mathrm{id},0)} \\
\mathbb{Q}\overline{\nabla}_S & \xrightarrow{\phi} & \mathbb{Q}(E_S \oplus \mathbb{Z}G \oplus C_0) \\
\downarrow & & \downarrow {\scriptstyle (\mathrm{id},u_0,\mathrm{id}_{C_0})} \\
\mathbb{Q}\overline{\nabla}_{S_0} & \xrightarrow{\phi_0} & \mathbb{Q}(E_{S_0} \oplus C_0)
\end{array} \tag{22}$$

Here, the two columns derive from the sequences (19) and (20). Note that the second map in (20) has finite cokernel. We are ready to prove

Theorem 3.2. *Fix a set of data (D). Let $\mathfrak{P}_0$ be a prime not in S which ramifies in L/K and h an integral multiple of h_L, the class number of L. Assume that there is a $\mathbb{Q}G$-isomorphism ϕ_0 that fits into diagram (22).Then we have an equality*

$$\Omega_{\phi_0} - \Omega_\phi = \partial[\operatorname{ind}_{G_{\mathfrak{P}_0}}^G \mathbb{Q}, -h|G_{\mathfrak{P}_0}|].$$

In particular, $\Omega_{\phi_0} - \Omega_\phi$ has representing homomorphism

$$\chi \mapsto (-h|G_{\mathfrak{P}_0}|)^{\dim V_\chi^{G_{\mathfrak{P}_0}}},$$

where V_χ is a $\mathbb{C}G$-module with character χ.

Proof. □

To complete this paragraph, we have to discuss how Ω_ϕ varies if S is enlarged by the orbit of a non-ramified prime $\mathfrak{P}_0$. As before let $S_0 := S \cup G\mathfrak{P}_0$. The exact sequence (12) for the sets S and S_0 together with the natural exact sequence $\mathbb{Z}S \rightarrowtail \mathbb{Z}S_0 \twoheadrightarrow \operatorname{ind}_{G_{\mathfrak{P}_0}}^G \mathbb{Z}$ yield an exact sequence

$$\overline{\nabla}_S \rightarrowtail \overline{\nabla}_{S_0} \twoheadrightarrow \operatorname{ind}_{G_{\mathfrak{P}_0}}^G \mathbb{Z}.$$

For each finite prime $\mathfrak{P}$ of L let us write $v_\mathfrak{P}$ for the normalized valuation at $\mathfrak{P}$. The map

$$E_{S_0} \to \mathbb{Z}[G/G_{\mathfrak{P}_0}] = \operatorname{ind}_{G_{\mathfrak{P}_0}}^G \mathbb{Z},\ u \mapsto \sum_{g \in G/G_{\mathfrak{P}_0}} v_{\mathfrak{P}_0}(g \cdot u) g^{-1}$$

has kernel E_S and finite cokernel. Thus, for each isomorphism $\phi : \mathbb{Q}\overline{\nabla}_S \to \mathbb{Q}(E_S \oplus C)$, where C is $\mathbb{Z}G$-free of rank $|S_{\mathrm{ram}}^* \setminus (S \cap S_{\mathrm{ram}})^*|$, there is an isomorphism ϕ_0 fitting in a commutative diagram

$$\begin{array}{ccc} \mathbb{Q}\overline{\nabla}_S & \xrightarrow{\ \phi\ } & \mathbb{Q}(E_S \oplus C) \\ \downarrow & & \downarrow \\ \mathbb{Q}\overline{\nabla}_{S_0} & \xrightarrow{\ \phi_0\ } & \mathbb{Q}(E_{S_0} \oplus C) \\ \downarrow & & \downarrow \\ \operatorname{ind}_{G_{\mathfrak{P}_0}}^G \mathbb{Q} & = & \operatorname{ind}_{G_{\mathfrak{P}_0}}^G \mathbb{Q} \end{array} \tag{23}$$

The result corresponding to Theorem 3.2 is exactly the same as for large sets S (cf. [GRW99], p. 60):

Theorem 3.3. *Fix a set of data (D) and let $\mathfrak{P}_0$ be a prime not in S which does not ramify in L/K. Given a $\mathbb{Q}G$-isomorphism ϕ_0 that fits in diagram (23) we have an equality*

$$\Omega_{\phi_0} - \Omega_\phi = \partial[\mathbb{Q}G, \eta].$$

Here, η is the $\mathbb{Q}G$-automorphism given by

$$\eta(1) = |G_{\mathfrak{P}_0}|\varepsilon_0 + \frac{1}{|G_{\mathfrak{P}_0}|} \sum_{i=0}^{|G_{\mathfrak{P}_0}|-1} i\phi_{\mathfrak{P}_0}^i (1 - \varepsilon_0),$$

where $\varepsilon_0 = \frac{1}{|G_{\mathfrak{P}_0}|} N_{G_{\mathfrak{P}_0}}$ and $\phi_{\mathfrak{P}_0}$ is the Frobenius automorphism at $\mathfrak{P}_0$.
In particular, $\Omega_{\phi_0} - \Omega_\phi$ has representing homomorphism

$$\chi \mapsto (|G_{\mathfrak{P}_0}|)^{\dim V_\chi^{G_{\mathfrak{P}_0}}} \cdot \det(\phi_{\mathfrak{P}_0} - 1 | V_\chi / V_\chi^{G_{\mathfrak{P}_0}})^{-1},$$

where V_χ is a $\mathbb{C}G$-module with character χ.

The proof is similar to (and indeed easier than) the proof of Theorem 3.2 and left to the reader. But see [Ni08], Theorem 1.4.4.

4. The conjecture

Thanks to the results of the last paragraph we are now able to state a LRNC for small sets of places. But before doing so we recall the basic ingredients of this conjecture apart from Ω_ϕ.
So let us fix a finite Galois extension L/K of number fields with Galois group G and a finite G-invariant set S of places of L, which contains all the archimedean primes. Then there are $\mathbb{Q}G$-isomorphisms

$$\phi : \Delta_{\mathbb{Q}} S \xrightarrow{\simeq} \mathbb{Q}E_S,$$

and the Stark-Tate regulator is defined to be

$$\begin{array}{rcl} R_\phi : R(G) & \to & \mathbb{C}^\times \\ \chi & \mapsto & \det(\lambda_S \phi | \mathrm{Hom}_G(V_{\check{\chi}}, \Delta_{\mathbb{C}} S)), \end{array}$$

where λ_S is the Dirichlet map (**??**) and $V_{\check{\chi}}$ is a $\mathbb{C}G$-module whose character is contragredient to χ. One defines

$$\begin{array}{rcl} A_\phi : R(G) & \to & \mathbb{C}^\times \\ \chi & \mapsto & R_\phi(\chi)/c_S(\chi). \end{array}$$

Let $\mathbb{Q}^c$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. There is the following conjecture of Stark:

Conjecture 4.1 (Stark). $A_\phi(\chi) \in \mathbb{Q}^c$ and $A_\phi(\chi^\sigma) = A_\phi(\chi)^\sigma$ for all $\sigma \in \mathrm{Gal}(\mathbb{Q}^c/\mathbb{Q})$.

Alternatively, one can choose a number field $F \subset \mathbb{C}$, Galois over $\mathbb{Q}$ with Galois group Γ, which is large enough such that all representations of G can be realized over F. Then conjecture 4.1 is equivalent to $A_\phi(\chi) \in F$ and $A_\phi(\chi^\sigma) = A_\phi(\chi)^\sigma$ for all $\sigma \in \Gamma$, i.e. $A_\phi \in \mathrm{Hom}_\Gamma(R(G), F^\times)$.
Let us denote by $W(\chi)$ the Artin root number of the character χ. Then it holds (cf. [We96], Prop. 7(b), p.57):

Proposition 4.2. *If χ is an irreducible symplectic character of G, then $A_\phi(\chi)W(\chi) \in \mathbb{R}^+$.*

Denote the infinite prime of the embedding $F \subset \mathbb{C}$ by $\wp_\infty$. Define $W(L/K, \cdot) \in \mathrm{Hom}_\Gamma(R(G), J_F)$ by

$$W(L/K, \chi)_\wp = \begin{cases} W(\chi^{\gamma^{-1}}) & \text{if } \chi \text{ is symplectic and } \wp = \wp_\infty^\gamma \\ 1 & \text{otherwise} \end{cases}$$

such that the homomorphism $\chi \mapsto A_\phi(\chi)W(L/K, \chi)$ lies in $\mathrm{Hom}_\Gamma^+(R(G), J_F)$ if Stark's conjecture holds. For large S the LRNC states

Conjecture 4.3 (LRNC for large S). The element $\Omega_\phi \in K_0T(\mathbb{Z}G)$ has representing homomorphism $\chi \mapsto A_\phi(\check{\chi})W(L/K, \check{\chi})$.

In the construction of Ω_ϕ for small sets S, the module ΔS has been replaced by $\overline{\nabla}_S$. We aim to define a modified Dirichlet map

$$\lambda_S^{\mathrm{mod}} : E_S \oplus C \longrightarrow \mathbb{R} \otimes \overline{\nabla}_S,$$

where C is a free $\mathbb{Z}G$-module of rank $|S_{\mathrm{ram}}^* \setminus (S \cap S_{\mathrm{ram}})^*|$. For this, we have to take a closer look at the modules $W_{\mathfrak{P}}^0$, especially for ramified primes $\mathfrak{P}$.

Let us write $\phi_{\mathfrak{P}}$ for the Frobenius automorphism at $\mathfrak{P}$ as well as for a fixed lift of it. The inertial lattice $W_{\mathfrak{P}}$ is the kernel of the map

$$\begin{array}{rcl} \Delta G_{\mathfrak{P}} \times \mathbb{Z}\overline{G_{\mathfrak{P}}} & \longrightarrow & \mathbb{Z}\overline{G_{\mathfrak{P}}} \\ (g-1, \overline{h}) & \mapsto & \overline{g} - 1 + (1 - \phi_{\mathfrak{P}})\overline{h}. \end{array}$$

Hence, using the identifications concerning $\mathbb{Z}$-duals explained in the preliminaries, we achieve a description of $W_{\mathfrak{P}}^0$ as the cokernel of the map (cf. [Gr07], §5)

$$\begin{array}{rcl} \mathbb{Z}\overline{G_{\mathfrak{P}}} & \longrightarrow & \mathbb{Z}G_{\mathfrak{P}}/N_{G_{\mathfrak{P}}} \times \mathbb{Z}\overline{G_{\mathfrak{P}}} \\ 1 & \mapsto & (N_{I_{\mathfrak{P}}}, 1 - \phi_{\mathfrak{P}}^{-1}). \end{array}$$

Proposition 4.4. *Let κ denote the canonical epimorphism from $(\mathbb{Z}G_{\mathfrak{P}})^2$ onto $W_{\mathfrak{P}}^0$ and define*

$$\begin{array}{rcl} q : W_{\mathfrak{P}} & \longrightarrow & (\mathbb{Z}G_{\mathfrak{P}})^2 \\ (x, y) & \mapsto & (N_{I_{\mathfrak{P}}}\phi_{\mathfrak{P}} \cdot y, x). \end{array}$$

Then it holds:

i) *The kernel of κ is generated by $z = (N_{I_{\mathfrak{P}}}, 1 - \phi_{\mathfrak{P}}^{-1})$ and $0 \times \Delta(G_{\mathfrak{P}}, \overline{G_{\mathfrak{P}}})$, where $\Delta(G_{\mathfrak{P}}, \overline{G_{\mathfrak{P}}})$ is the kernel of the canonical projection $\mathbb{Z}G_{\mathfrak{P}} \twoheadrightarrow \mathbb{Z}\overline{G_{\mathfrak{P}}}$.*

ii) *The diagram*

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{1 \mapsto N_{G_{\mathfrak{P}}}} & \mathbb{Z}G_{\mathfrak{P}} & \twoheadrightarrow & \mathbb{Z}G_{\mathfrak{P}}/N_{G_{\mathfrak{P}}} \\ \downarrow & & \downarrow \iota_1 & & \downarrow \iota_1 \\ W_{\mathfrak{P}} & \xrightarrow{q} & (\mathbb{Z}G_{\mathfrak{P}})^2 & \xrightarrow{\kappa} & W_{\mathfrak{P}}^0 \\ \downarrow \mathrm{pr}_x & & \downarrow \mathrm{pr}_2 & & \downarrow (0, \mathrm{aug}_{\overline{G_{\mathfrak{P}}}}) \\ \Delta G_{\mathfrak{P}} & \hookrightarrow & \mathbb{Z}G_{\mathfrak{P}} & \twoheadrightarrow & \mathbb{Z} \end{array}$$

commutes and has exact rows and columns.

Proof. □

We now set

$$d_{\mathfrak{P}} := \frac{1}{|G_{\mathfrak{P}}|}\kappa(|G_{\mathfrak{P}}|, N_{G_{\mathfrak{P}}}) \in \mathbb{Q}W_{\mathfrak{P}}^0.$$

Observe that this definition differs from the corresponding element $d_{\mathfrak{p}}$ in [Gr07].

Lemma 4.5. *$d_{\mathfrak{P}}$ is a $\mathbb{Q}G_{\mathfrak{P}}$-generator of $\mathbb{Q}W_{\mathfrak{P}}^0$.*

Proof. □

Let $1_{\mathfrak{P}}$, $\mathfrak{P} \in S_{\mathrm{ram}}^* \setminus (S \cap S_{\mathrm{ram}})^*$ be a $\mathbb{Z}G$-basis of the free $\mathbb{Z}G$-module C. We choose a positive multiple h of h_L and $u_{\mathfrak{P}} \in L$ such that $v_{\mathfrak{P}}(u_{\mathfrak{P}}) = h$ and $v_{\mathfrak{Q}}(u_{\mathfrak{P}}) = 0$ for all finite primes $\mathfrak{Q} \neq \mathfrak{P}$. We define

$$\begin{array}{rcl} \lambda_C : C & \longrightarrow & \mathbb{R} \otimes \bigoplus_{\mathfrak{P} \in S_{\mathrm{ram}}^* \setminus (S \cap S_{\mathrm{ram}})^*} \mathrm{ind}_{G_{\mathfrak{P}}}^G W_{\mathfrak{P}}^0 \oplus \mathbb{R}S_\infty \\ 1_{\mathfrak{P}} & \mapsto & \left(h \log N(\mathfrak{P}) \frac{1}{|G_{\mathfrak{P}}|} N_{G_{\mathfrak{P}}} + 1 - \frac{1}{|G_{\mathfrak{P}}|} N_{G_{\mathfrak{P}}}\right) d_{\mathfrak{P}} - \sum_{\mathfrak{Q} | \infty} \log |u_{\mathfrak{P}}|_{\mathfrak{Q}} \mathfrak{Q}. \end{array}$$

By the second part of Proposition 4.4 we have

$$(0, \mathrm{aug}_{\overline{G_{\mathfrak{P}}}})(d_{\mathfrak{P}}) = \mathrm{aug}\,(\mathrm{pr}_2(1, \frac{1}{|G_{\mathfrak{P}}|} N_{G_{\mathfrak{P}}})) = 1.$$

Hence, the projection in sequence (12) sends $\lambda_C(1_{\mathfrak{P}})$ to

$$h \log N(\mathfrak{P}) - \sum_{\mathfrak{Q}|\infty} \log |u_{\mathfrak{P}}|_{\mathfrak{Q}} = - \sum_{\text{all } \mathfrak{Q}} \log |u_{\mathfrak{P}}|_{\mathfrak{Q}} = 0.$$

Thus, the image of λ_C lies in $\mathbb{R}\overline{\nabla}$, and we may define a modified Dirichlet map by

$$\begin{array}{rcl} \lambda_S^{\mathrm{mod}} : E_S \oplus C & \longrightarrow & \mathbb{R}\overline{\nabla} \\ (e, c) & \mapsto & \lambda_S(e) + \lambda_C(c), \end{array} \tag{24}$$

where λ_S is the usual Dirichlet map (**??**). Note that λ_S^{mod} depends on the choices of h and the $u_{\mathfrak{P}}$.

Definition 4.6. We call the map

$$\begin{array}{rcl} R_\phi^{\mathrm{mod}} : R(G) & \longrightarrow & \mathbb{C}^\times \\ \chi & \mapsto & \dfrac{\det(\lambda_S^{\mathrm{mod}} \phi | \mathrm{Hom}_G(V_{\check{\chi}}, \mathbb{C}\overline{\nabla}_S))}{\prod_{\mathfrak{P} \in S_{\mathrm{ram}}^* \setminus (S \cap S_{\mathrm{ram}})^*} (-h|G_{\mathfrak{P}}|)^{\dim V_{\check{\chi}}^{G_{\mathfrak{P}}}}} \end{array}$$

the **modified Stark-Tate regulator** and set

$$\begin{array}{rcl} A_\phi^{\mathrm{mod}} : R(G) & \longrightarrow & \mathbb{C}^\times \\ \chi & \mapsto & \dfrac{R_\phi^{\mathrm{mod}}(\chi)}{c_{S \cup S_{\mathrm{ram}}}(\chi)}. \end{array}$$

Remark 3. If the set S already contains all the ramified primes, we obviously have $R_\phi^{\mathrm{mod}} = R_\phi$ and $A_\phi^{\mathrm{mod}} = A_\phi$.

Unfortunately, the above definition is not independent of the choices of h and the $u_{\mathfrak{P}}$. Nevertheless, we have the following

Proposition 4.7. *The maps R_ϕ^{mod}, $A_\phi^{\mathrm{mod}} \in \mathrm{Hom}(R(G), \mathbb{C}^\times)$ are well defined.*

Proof. □

The properties of the homomorphism A_ϕ^{mod} are summarized in the following

Theorem 4.8. *Fix a set of data (D). Let $F \subset \mathbb{C}$ be a number field, Galois over $\mathbb{Q}$ with Galois group Γ, which is large enough such that all representations of G can be realized over F. Then the following holds:*

i) *$A_\phi^{\mathrm{mod}}(\chi) \in F$ and $A_\phi^{\mathrm{mod}}(\chi^\sigma) = A_\phi^{\mathrm{mod}}(\chi)^\sigma$ for all $\sigma \in \Gamma$ if and only if Stark's conjecture (4.1) holds.*

ii) *If χ is an irreducible symplectic character of G, then $A_\phi^{\mathrm{mod}}(\chi) W(\chi) \in \mathbb{R}^+$.*

iii) *If $\phi' : \mathbb{Q}\overline{\nabla} \to \mathbb{Q}(E_S \oplus C)$ is another $\mathbb{Q}G$-isomorphism, then*

$$\frac{A_{\phi'}^{\mathrm{mod}}(\chi)}{A_\phi^{\mathrm{mod}}(\chi)} \equiv \det(\phi^{-1}\phi' | \mathrm{Hom}_G(V_{\check{\chi}}, \mathbb{C}\overline{\nabla})) \bmod \mathrm{Det}\,(U(\mathbb{Z}G)).$$

iv) *Let $\mathfrak{P}_0$ be a prime not in S which ramifies in L/K. Given an integral multiple h of h_L, the class number of L, and $\mathbb{Q}G$-isomorphisms ϕ and ϕ_0 as in diagram (22) we have an equality*

$$\frac{A_{\phi_0}^{\mathrm{mod}}(\chi)}{A_\phi^{\mathrm{mod}}(\chi)} \equiv (-h|G_{\mathfrak{P}_0}|)^{\dim V_{\check{\chi}}^{G_{\mathfrak{P}_0}}} \bmod \mathrm{Det}\,(U(\mathbb{Z}G)).$$

v) *Let $\mathfrak{P}_0$ be a prime not in S which does not ramify in L/K. Given $\mathbb{Q}G$-isomorphisms ϕ and ϕ_0 as in diagram (23) we have an equality*

$$\frac{A_{\phi_0}^{\mathrm{mod}}(\chi)}{A_{\phi}^{\mathrm{mod}}(\chi)} \equiv (|G_{\mathfrak{P}_0}|)^{\dim V_{\check{\chi}}^{G_{\mathfrak{P}_0}}} \cdot \det(\phi_{\mathfrak{P}_0} - 1 | V_{\check{\chi}}/V_{\check{\chi}}^{G_{\mathfrak{P}_0}})^{-1} \bmod \mathrm{Det}\,(U(\mathbb{Z}G)).$$

Before proving the theorem, we now point out how to state a LRNC for small sets of places. Assume that Stark's conjecture holds. By (i), (ii) and Proposition 4.7 we can view the map

$$\chi \mapsto A_{\phi}^{\mathrm{mod}}(\check{\chi})W(L/K, \check{\chi})$$

as a representing homomorphism of an element in $K_0(\mathbb{Z}G, \mathbb{Q})$ via the isomorphisms (5) and (9). Since Theorem 4.8 together with Proposition 3.1, Theorem 3.2 and Theorem 3.3 show that this homomorphism exactly behaves like Ω_ϕ, it is now evident to state the

Conjecture 4.9 (LRNC for small S). The element $\Omega_\phi \in K_0(\mathbb{Z}G, \mathbb{Q})$ has representing homomorphism $\chi \mapsto A_{\phi}^{\mathrm{mod}}(\check{\chi})W(L/K, \check{\chi})$.

Theorem 4.8 now implies the

Corollary 4.10. *The Lifted Root Number Conjecture for small sets of places is equivalent to the Lifted Root Number Conjecture for large sets of places.*

For this reason we refer to conjecture 4.9 as well as to conjecture 4.3 as the LRNC.
The element Ω_ϕ decomposes into p-parts $\Omega_\phi^{(p)}$ via the isomorphism (7). If we choose a prime $\wp$ in F above p and an embedding $j_p : F \rightarrowtail F_\wp$ for each p, Stark's conjecture asserts that the map

$$(A_{\phi}^{\mathrm{mod}})^{(p)} : \chi \mapsto j_p(A_{\phi}^{\mathrm{mod}}(j_p^{-1}(\chi)))$$

lies in $\mathrm{Hom}_{\Gamma_\wp}(R_p(G), F_\wp^\times)$. Conjecture 4.9 localizes to

Conjecture 4.11 (LRNC for small S at the prime p). The element $\Omega_\phi^{(p)} \in K_0(\mathbb{Z}_pG, \mathbb{Q}_p)$ has representing homomorphism $\chi \mapsto (A_{\phi}^{\mathrm{mod}})^{(p)}(\check{\chi})$.

We obviously have the

Corollary 4.12. *The Lifted Root Number Conjecture is true for L/K if and only if Conjecture 4.11 is true for L/K and all primes p.*

We conclude this section with the

Proof of Theorem 4.8. □

5. An exercise: Nice extensions

The aim of this section is to lift a result of C. Greither [Gr00] on Chinburg's Ω_3-conjecture.
If L/K is an abelian CM-extension with Galois group G, we denote by j the unique automorphism of L induced by complex conjugation. A character χ of G is called odd (resp. even) if $\chi(j) = -1$ (resp. $\chi(j) = 1$). Note that for odd primes p the LRNC naturally decomposes in a plus and a minus part which corresponds to the even and odd characters, respectively. Let μ_L be the roots of unity in L, and L^{cl} the Galois closure of L over $\mathbb{Q}$; it is easy to see that L^{cl} is again a CM-field. In loc.cit. a CM-extension L/K is called *nice* if the following holds:

i) L/K is an abelian CM-extension with Galois group G

ii) The complex conjugation $j \in G$ lies in the decomposition group $G_{\mathfrak{P}}$ for all primes $\mathfrak{P}$ which ramify in L/K

iii) If p is an odd prime such that $L^{\mathrm{cl}} \subset L^{\mathrm{cl},+}(\zeta_p)$ then $j \in G_{\mathfrak{P}}$ for all primes $\mathfrak{P}$ above p.

iv) $\mu_L \otimes \mathbb{Z}_p$ is c.t. for all odd primes p.

THEOREM 5.1. *Let L/K be a nice CM-extension. Then the minus part of the LRNC at p holds for all odd primes p.*

Proof. □

Remark 4. A wider application of the LRNC for small sets S is given in [Ni].

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