

NOTES ON NONCOMMUTATIVE FITTING INVARIANTS

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WITH AN APPENDIX BY HENRI JOHNSTON AND ANDREAS NICKEL

ABSTRACT. To each finitely presented module M over a commutative ring R one can associate an R -ideal $\text{Fitt}_R(M)$, which is called the (zeroth) Fitting ideal of M over R . This is of interest because it is always contained in the R -annihilator $\text{Ann}_R(M)$ of M , but is often much easier to compute. This notion has recently been generalised to that of so-called ‘Fitting invariants’ over certain noncommutative rings; the present author considered the case in which R is an \mathfrak{o} -order Λ in a finite dimensional separable algebra, where \mathfrak{o} is an integrally closed commutative noetherian complete local domain. This article is a survey of known results and open problems in this context. In particular, we investigate the behaviour of Fitting invariants under direct sums. In the appendix, we present a new approach to Fitting invariants via Morita equivalence.

INTRODUCTION

Let R be a commutative unitary ring and let M be a finitely presented R -module. This means that there is an exact sequence

$$(0.1) \quad R^a \xrightarrow{h} R^b \longrightarrow M \longrightarrow 0,$$

where a and b are positive integers. In other words, the module M is finitely generated and there is a finite number of relations between the generators. This information is incorporated in the $a \times b$ matrix h . Note that every finitely generated module over a noetherian ring is indeed finitely presented.

Suppose that $a \geq b$. Then the (zeroth) Fitting ideal of M over R is defined to be the R -ideal generated by all $b \times b$ minors of the matrix h :

$$\text{Fitt}_R(M) := \langle \det(H) \mid H \in S_b(h) \rangle_R,$$

where $S_b(h)$ denotes the set of all $b \times b$ submatrices of h . In the case $a < b$ one simply puts $\text{Fitt}_R(M) := 0$. This notion was introduced by the German mathematician Hans Fitting [Fit36] who showed that it is in fact independent of the chosen finite presentation h . Fitting was a student of Emmy Noether and is also famous for his contributions to group theory.

Fitting ideals became an important tool in commutative algebra. A key property is that the Fitting ideal of M is always contained in the R -annihilator ideal of M , but in many cases is much easier to compute. For instance, it behaves well under epimorphisms, certain exact sequences, and direct sums of R -modules: if M and N are two finitely presented R -modules, then one has an equality

$$(0.2) \quad \text{Fitt}_R(M \oplus N) = \text{Fitt}_R(M) \cdot \text{Fitt}_R(N).$$

For a full account of the theory, we refer the reader to [Nor76].

Fitting ideals have many applications in number theory. We give two typical examples. If L/K is a finite Galois extension of number fields with Galois group G , then the class

Date: Version of 10th September 2018.

group cl_L of L has a natural structure as a module over the group ring $\mathbb{Z}[G]$. If p is a prime then the p -part $\mathbb{Z}_p \otimes_{\mathbb{Z}} \text{cl}_L$ of the class group is a module over the p -adic group ring $\mathbb{Z}_p[G]$.

Now assume that p is odd and that L/K is a CM-extension. If G is abelian then Greither [Gre07] has computed the Fitting ideal of the Pontryagin dual of the minus part of $\mathbb{Z}_p \otimes_{\mathbb{Z}} \text{cl}_L$ via the equivariant Tamagawa number conjecture. This gives strong evidence for a conjecture of Brumer that asserts that certain ‘Stickelberger elements’ (constructed from values at zero of Artin L -functions attached to the irreducible characters of G) annihilate the class group.

Fitting ideals also appear in Iwasawa theory. The formulation and the proof of the (classical) main conjecture for totally real fields by Wiles [Wil90] makes heavy use of the close relation between characteristic ideals and Fitting ideals of Iwasawa modules.

Fitting ideals of arithmetic objects are also interesting in their own right. Kurihara and Miura [KM11] showed that (away from the 2-primary part) the Fitting ideal of the minus class group of an absolutely abelian imaginary number field coincides with the Stickelberger ideal (as conjectured by Kurihara [Kur03]). This gives the precise arithmetical interpretation of the latter ideal.

As Galois groups are in general non-abelian, it is natural to ask whether analogous invariants can be defined for modules over noncommutative rings such as p -adic group rings. Grime considered several cases in his PhD thesis [Gri02], including matrix rings over commutative rings. We will describe an approach to noncommutative Fitting invariants over rings that are Morita equivalent to a commutative ring in the appendix. This is essentially a generalisation of Grime’s approach (and of [JN13, §2]). Parker has treated the case of p -adic group rings in his PhD thesis [Par07] under the (for applications sometimes too restrictive) hypothesis that one can choose $a = b$ in (0.1).

Now let \mathfrak{o} be an integrally closed commutative noetherian complete local domain with field of quotients F . Let Λ be an \mathfrak{o} -order in a finite dimensional separable algebra A over F . We will call such an order a Fitting order over \mathfrak{o} . A standard example is that of p -adic group rings $\mathbb{Z}_p[G]$ where p is a prime and G is a finite group. The Iwasawa algebra $\mathbb{Z}_p[[G]]$ of a one-dimensional p -adic Lie group G is a second example.

Let Λ be an arbitrary Fitting order and let M be a finitely presented Λ -module. In [Nic10] the present author defined the ‘maximal Fitting invariant’ $\text{Fitt}_{\Lambda}^{\max}(M)$ of M over Λ as an equivalence class of certain modules over the centre of Λ using reduced norms. If Λ is commutative then the reduced norm coincides with the usual determinant and thus this notion is compatible with that of the classical Fitting ideal. This approach has been studied further by Johnston and the present author in [JN13]. It has also been applied in number theory to study the class group and other Galois modules in extensions with arbitrary Galois groups [Nic11b, Nic13, Nic11a, BS17, BMCW18, JN] (see also [Nic] for a survey).

In this article we do not use the notion of ‘ $\text{Nrd}(\Lambda)$ -equivalence’ as in [Nic10]. We essentially follow the alternative approach in [JN13, §3.5] and define Fitting invariants as a genuine ideal of a certain commutative ring $\mathcal{I}(\Lambda)$ that contains the centre of Λ and the reduced norms of every matrix with entries in Λ . As long as we are only interested in annihilation results, this approach has no disadvantage over the more complicated notion of $\text{Nrd}(\Lambda)$ -equivalence. The latter is only necessary when one wishes to relate Fitting invariants to (relative) K -theory.

In order to obtain annihilators from $\text{Fitt}_{\Lambda}^{\max}(M)$ one has to multiply by a certain ideal in $\mathcal{I}(\Lambda)$ which we call the denominator ideal. In this article we report on basic properties

of noncommutative Fitting invariants and on lower bounds for the denominator ideal. In particular, we consider the case where Λ is a p -adic group ring. Inspired by [BG03, Lemma 6] and recent work of Kataoka [Kat18], we give a new shorter and simpler proof of a proposition in [Nic10] on the behaviour of Fitting invariants under Pontryagin duality. Finally, in section §5 we investigate whether Fitting invariants are additive in the sense of (0.2) when working over noncommutative rings. We show that this property does hold for certain classes of Fitting orders but, perhaps surprisingly, that it does not hold for non-maximal hereditary Fitting orders over complete discrete valuation rings.

In the appendix, we present a new approach to Fitting invariants for unitary rings Λ which are Morita equivalent to a commutative ring. This generalises [JN13, §2], where the case of matrix rings over commutative rings is considered.

In many cases we will not provide rigorous proofs. Instead we try to motivate the results and give many examples.

Acknowledgements. The author acknowledges financial support provided by the Deutsche Forschungsgemeinschaft (DFG) within the Heisenberg programme (No. NI 1230/3-1). I am indebted to Masato Kurihara, Kenichi Bannai and Takeshi Tsuji for the excellent organisation of the Iwasawa 2017 conference, where I had the opportunity to give a Preparatory Lecture Series on ‘Non-abelian Stark-type conjectures and noncommutative Iwasawa theory’ including most of the material presented in this article. I also thank Henri Johnston, Takenori Kataoka and David Watson for fruitful discussions and help concerning various aspects of Fitting invariants.

Notation and conventions. All rings are assumed to have an identity element and all modules are assumed to be left modules unless otherwise stated. If Λ is a ring, we write $M_{m \times n}(\Lambda)$ for the set of all $m \times n$ matrices with entries in Λ . We denote the group of invertible matrices in $M_{n \times n}(\Lambda)$ by $\mathrm{GL}_n(\Lambda)$ and let $\mathbf{1}_n \in \mathrm{GL}_n(\Lambda)$ be the $n \times n$ identity matrix. Moreover, we let $\zeta(\Lambda)$ denote the centre of the ring Λ . We shall sometimes abuse notation by using the symbol \oplus to denote the direct product of rings or orders.

1. THE COMMUTATIVE CASE

The material presented in this section originates with Fitting [Fit36]. We refer the reader to [Nor76] and [Eis95, §20] for further details.

Let R be a commutative ring and let M be a finitely presented R -module. Choose a finite presentation h as in (0.1) and define

$$\mathrm{Fitt}_R(M) := \begin{cases} 0 & \text{if } a < b \\ \langle \det(H) \mid H \in S_b(h) \rangle_R & \text{if } a \geq b, \end{cases}$$

where we recall that $S_b(h)$ denotes the set of all $b \times b$ submatrices of h . We call $\mathrm{Fitt}_R(M)$ the *Fitting ideal* of M over R .

Theorem 1.1. *Let R be a commutative ring and let M be a finitely presented R -module. Then $\mathrm{Fitt}_R(M)$ is independent of the choice of h and thus is well-defined.*

The idea of the proof is as follows (for a full proof see [Nor76, §3.1] or [Eis95, §20.2]). Since two ideals are equal if and only if they become equal in every localisation of R , we can and do assume that R is local. We choose $b' \in \mathbb{N}$ minimal such that there is a surjection $R^{b'} \twoheadrightarrow M$. Similarly, we choose $a' \in \mathbb{N}$ minimal such that there is a finite presentation

$$R^{a'} \xrightarrow{h'} R^{b'} \longrightarrow M \longrightarrow 0.$$

It suffices to show that the Fitting ideals coming from h and h' coincide. We may view (0.1) as the truncation of a free resolution \mathcal{F} of M . Similarly, we may view h' as a truncation of a *minimal* free resolution \mathcal{F}' of M . However, as R is local, every free resolution is isomorphic to the direct sum of \mathcal{F}' and a trivial complex (see [Eis95, Theorem 20.2]). In particular, there are invertible matrices $X \in \mathrm{GL}_a(R)$ and $Y \in \mathrm{GL}_b(R)$ such that

$$h \circ X = Y \circ \begin{pmatrix} h' & 0 & 0 \\ 0 & \mathbf{1} & 0 \end{pmatrix},$$

where $\mathbf{1} = \mathbf{1}_{b-b'}$ is a $(b-b') \times (b-b')$ identity matrix. As $\det(Y)$ belongs to R^\times , we may assume that $Y = 1$. By an exercise in linear algebra that uses the multilinearity of the determinant we may likewise assume that $X = 1$. The result now follows easily.

Example 1.2. Let $I = \langle r_1, \dots, r_a \rangle_R$ be a finitely generated ideal of R and take $M = R/I$. Then we have a finite presentation

$$R^a \longrightarrow R \longrightarrow R/I \longrightarrow 0,$$

where the first arrow maps the i -th standard basis vector of R^a to r_i , $1 \leq i \leq a$. Thus we have that

$$\mathrm{Fitt}_R(R/I) = I.$$

Remark 1.3. Let $R \rightarrow S$ be a homomorphism of commutative rings. As taking tensor products is a right exact functor, a finite presentation (0.1) of the R -module M yields a finite presentation

$$S^a \longrightarrow S^b \longrightarrow S \otimes_R M \longrightarrow 0$$

of $S \otimes_R M$. Fitting ideals therefore commute with base change:

$$\mathrm{Fitt}_S(S \otimes_R M) = S \otimes_R \mathrm{Fitt}_R(M).$$

Remark 1.4. Let M be a finitely presented R -module with a finite presentation h as in (0.1). For each integer $i \geq 0$ one can define ‘higher Fitting ideals’ $\mathrm{Fitt}_R^i(M)$ of M that are generated by the minors of h of size $b-i$. These invariants form an increasing sequence

$$\mathrm{Fitt}_R(M) = \mathrm{Fitt}_R^0(M) \subseteq \mathrm{Fitt}_R^1(M) \subseteq \mathrm{Fitt}_R^2(M) \subseteq \dots$$

of R -ideals. Moreover, if M can be generated by q elements, then $\mathrm{Fitt}_R^q(M) = R$. See [Nor76, §3, Theorem 2] for a proof. If R is a principal ideal domain, then the higher Fitting ideals $\mathrm{Fitt}_R^i(M)$ for all $i \geq 0$ determine the R -module M up to isomorphism. This can be deduced from the structure theorem of finitely generated modules over principal ideal domains (see [Kur03, §1.1], for instance).

Let us denote the R -annihilator ideal of M by $\mathrm{Ann}_R(M)$. The main interest in Fitting ideals comes from the following fact.

Theorem 1.5. *Let R be a commutative ring and let M be a finitely presented R -module. Then one has an inclusion*

$$\mathrm{Fitt}_R(M) \subseteq \mathrm{Ann}_R(M).$$

Proof. Choose a finite presentation (0.1) of M . Let $H \in S_b(h)$ be a submatrix. As M is a homomorphic image of the cokernel of H , we may assume that $h = H$ and thus $a = b$.

Let $H^* \in M_{b \times b}(R)$ be the adjoint matrix of H . Then $H^*H = HH^* = \det(H)\mathbf{1}_b$ and so the result follows from the commutative diagram

$$\begin{array}{ccccc} R^b & \xrightarrow{H} & R^b & \longrightarrow & M \\ & \searrow^{H^*} & \downarrow \det(H) & & \downarrow \det(H) \\ R^b & \xrightarrow{H} & R^b & \longrightarrow & M \end{array}$$

once one notes that the right vertical arrow is zero. \square

Example 1.6. Let $R = \mathbb{Z}$ and let M be an arbitrary finitely generated \mathbb{Z} -module. By the fundamental theorem of finitely generated abelian groups there are unique integers $f, n \geq 0$ and positive integers $1 < d_1 \mid d_2 \mid \cdots \mid d_n$ such that

$$M \simeq \mathbb{Z}^f \oplus \bigoplus_{i=1}^n \mathbb{Z}/d_i\mathbb{Z}.$$

If $f > 0$ then clearly $\text{Fitt}_{\mathbb{Z}}(M) = 0$. If $f = 0$ we can take for h the diagonal matrix with entries d_1, \dots, d_n . It follows that

$$\text{Fitt}_{\mathbb{Z}}(M) = \begin{cases} 0 & \text{if } f > 0 \\ (\prod_{i=1}^n d_i)\mathbb{Z} & \text{if } f = 0. \end{cases}$$

Moreover, we clearly have

$$\text{Ann}_{\mathbb{Z}}(M) = \begin{cases} 0 & \text{if } f > 0 \\ d_n\mathbb{Z} & \text{if } f = 0. \end{cases}$$

In particular, the inclusion $\text{Fitt}_{\mathbb{Z}}(M) \subseteq \text{Ann}_{\mathbb{Z}}(M)$ is proper if and only if $f = 0$ and $n > 1$. Of course, similar observations hold for every principal ideal domain R .

Example 1.7. Let G be a finite abelian group and let ΔG be the kernel of the natural augmentation map $\mathbb{Z}[G] \rightarrow \mathbb{Z}$ that sends each $g \in G$ to 1. It is straightforward to show that

$$\text{Ann}_{\mathbb{Z}[G]}(\Delta G) = N_G\mathbb{Z},$$

where $N_G := \sum_{g \in G} g$ (see [Neu11, Satz 1.3]). By Theorem 1.5 we must have

$$\text{Fitt}_{\mathbb{Z}[G]}(\Delta G) = mN_G\mathbb{Z}$$

for some integer m . We now apply Remark 1.3 with $R = \mathbb{Z}[G]$ and $S = \mathbb{Z}$ so that

$$\text{Fitt}_{\mathbb{Z}}(\mathbb{Z} \otimes_{\mathbb{Z}[G]} \Delta G) = m|G|\mathbb{Z}.$$

Moreover, we have isomorphisms of abelian groups $\mathbb{Z} \otimes_{\mathbb{Z}[G]} \Delta G \simeq \Delta G/(\Delta G)^2 \simeq G$ so that

$$\text{Fitt}_{\mathbb{Z}}(\mathbb{Z} \otimes_{\mathbb{Z}[G]} \Delta G) = |G|\mathbb{Z}$$

by Example 1.6. It follows that $m \in \mathbb{Z}^\times$ and thus

$$\text{Fitt}_{\mathbb{Z}[G]}(\Delta G) = N_G\mathbb{Z} = \text{Ann}_{\mathbb{Z}[G]}(\Delta G).$$

We now record some basic facts about Fitting ideals.

Lemma 1.8. *Let R be a commutative ring and let M_1, M_2, M_3 be finitely presented R -modules.*

- (i) *If $\pi : M_1 \twoheadrightarrow M_2$ is an epimorphism, then $\text{Fitt}_R(M_1) \subseteq \text{Fitt}_R(M_2)$.*
- (ii) *Fitting ideals behave well under direct sums:*

$$\text{Fitt}_R(M_1 \oplus M_3) = \text{Fitt}_R(M_1) \cdot \text{Fitt}_R(M_3).$$

(iii) If $M_1 \xrightarrow{\iota} M_2 \rightarrow M_3 \rightarrow 0$ is an exact sequence, then

$$\text{Fitt}_R(M_1) \cdot \text{Fitt}_R(M_3) \subseteq \text{Fitt}_R(M_2).$$

Proof. We only sketch the proof. Let $R^{a_1} \xrightarrow{h_1} R^{b_1} \xrightarrow{\pi_1} M_1 \rightarrow 0$ be a finite presentation of M_1 . Put $\pi_2 := \pi \circ \pi_1$. Then one may construct a finite presentation

$$R^{a_2} \xrightarrow{(h_1|*)} R^{b_1} \xrightarrow{\pi_2} M_2 \longrightarrow 0$$

of M_2 by adding more relations if necessary. This shows (i). For (iii) we may therefore assume that ι is injective. Let h_1 and h_3 be finite presentations of M_1 and M_3 , respectively. As in the proof of the horseshoe lemma (see [Wei94, Lemma 2.2.8], for instance) one can construct a finite presentation h_2 of M_2 of shape

$$\begin{pmatrix} h_1 & g \\ 0 & h_3 \end{pmatrix}.$$

If $M_2 = M_1 \oplus M_3$ one may additionally assume that $g = 0$. From this one can deduce (ii) and (iii). \square

Example 1.9. Let I_1, \dots, I_n be finitely generated ideals of R . Then it follows from Example 1.2 and Lemma 1.8(ii) that

$$\text{Fitt}_R \left(\bigoplus_{j=1}^n R/I_j \right) = \prod_{j=1}^n I_j.$$

Example 1.10. Let p be a prime and let $R = \mathbb{Z}_p[[T]]$ be the power series ring in one variable over \mathbb{Z}_p . Let M be a finitely generated torsion R -module. Then by the structure theorem for Iwasawa modules [NSW08, Theorem 5.3.8] there is a pseudo-isomorphism

$$\alpha : M \longrightarrow \bigoplus_{i=1}^s R/p^{m_i} \oplus \bigoplus_{j=1}^t R/F_j^{n_j},$$

where $s, t \geq 0$, $m_i, n_j \geq 1$ are integers and the F_j are distinguished irreducible polynomials. This means in particular that α becomes an isomorphism in every localisation of R at a prime ideal of height 1. The characteristic ideal $\text{Char}_R(M)$ of M is defined to be the R -ideal generated by $\prod_{i=1}^s p^{m_i} \cdot \prod_{j=1}^t F_j^{n_j}$.

Now assume that M contains no finite non-trivial submodule (that is α is injective). Then the projective dimension of M is at most 1 by [NSW08, Proposition 5.3.19(i)]. Since R is a local ring, every projective R -module is free and so there is a short exact sequence

$$0 \longrightarrow R^a \longrightarrow R^a \longrightarrow M \longrightarrow 0.$$

It follows that $\text{Fitt}_R(M)$ is a principal ideal. Since two principal ideals over R are equal if and only if they become equal in every localisation of R at a height 1 prime ideal, we have that

$$\text{Fitt}_R(M) = \text{Char}_R(M)$$

in this case (see [Kur12, Lemma 9.1], for instance).

2. NONCOMMUTATIVE FITTING INVARIANTS: BASIC PROPERTIES

2.1. Fitting domains and Fitting orders. We now introduce the class of rings for which we intend to define Fitting invariants. We recall that an F -algebra A over a field F is called *separable* if $E \otimes_F A$ is a semisimple E -algebra for every field extension E of F . If F is a perfect field, then every finite dimensional semisimple F -algebra is indeed separable (as follows from [CR81, Corollary 7.6]).

Definition 2.1. Let \mathfrak{o} be an integrally closed commutative noetherian complete local domain with field of quotients F . Then we call \mathfrak{o} a *Fitting domain*. Let A be a finite dimensional separable F -algebra and let Λ be an \mathfrak{o} -order in A . Then Λ is called a *Fitting order* over \mathfrak{o} .

Remark 2.2. Let Λ be a Fitting order over the Fitting domain \mathfrak{o} . Then Λ is noetherian and so every finitely generated Λ -module is in fact finitely presented.

Example 2.3. Any complete discrete valuation ring \mathfrak{o} is a Fitting domain. Conversely, any Fitting domain of Krull dimension 1 is a complete discrete valuation ring.

Example 2.4. For any Fitting domain \mathfrak{o} and any positive integer n , the ring $M_{n \times n}(\mathfrak{o})$ is a Fitting order over \mathfrak{o} .

Example 2.5. Let p be a prime and let G be a finite group. Then the ring of p -adic integers \mathbb{Z}_p is a Fitting domain and the group ring $\mathbb{Z}_p[G]$ is a Fitting order over \mathbb{Z}_p .

Example 2.6. More generally, let \mathfrak{o} be an arbitrary Fitting domain with field of quotients F and let G be a finite group. Then an \mathfrak{o} -order Λ in $A := F[G]$ is a Fitting order over \mathfrak{o} if and only if $|G|$ is invertible in F .

Example 2.7. Let G be a profinite group containing a finite normal subgroup H such that $G/H \simeq \Gamma$, where Γ is a pro- p group isomorphic to \mathbb{Z}_p . Note that G can be written as a semi-direct product $H \rtimes \Gamma$ and is a one-dimensional p -adic Lie group. The Iwasawa algebra of G over \mathbb{Z}_p is defined to be

$$\mathbb{Z}_p[[G]] = \varprojlim \mathbb{Z}_p[G/N],$$

where the inverse limit is taken over all open normal subgroups N of G . Since any homomorphism $\Gamma \rightarrow \text{Aut}(H)$ must have open kernel, we may choose a natural number n such that Γ^{p^n} is central in G . We put $\mathfrak{o} := \mathbb{Z}_p[[\Gamma^{p^n}]]$ and $F := \text{Quot}(\mathfrak{o})$. Then \mathfrak{o} is non-canonically isomorphic to the power series ring $\mathbb{Z}_p[[T]]$ in one variable over \mathbb{Z}_p and is thus a Fitting domain. If we view $\mathbb{Z}_p[[G]]$ as an \mathfrak{o} -module (or indeed as a left $\mathfrak{o}[H]$ -module), there is a decomposition

$$\mathbb{Z}_p[[G]] = \bigoplus_{i=0}^{p^n-1} \mathfrak{o}[H]\gamma^i,$$

where γ is a topological generator of Γ . This shows that $\mathbb{Z}_p[[G]]$ is a Fitting order over \mathfrak{o} in the separable F -algebra $A = \mathcal{Q}(G) := \bigoplus_i F[H]\gamma^i$.

2.2. Reduced norms and the integrality ring. Let \mathfrak{o} be a Fitting domain with field of quotients F and let A be a finite dimensional separable F -algebra. By Wedderburn's theorem A decomposes into

$$A = A_1 \oplus \cdots \oplus A_t,$$

where each A_i is isomorphic to an algebra of $n_i \times n_i$ matrices over a skewfield D_i . Then $F_i := \zeta(A_i) = \zeta(D_i)$ is a finite field extension of F and A_i is a central simple F_i -algebra. The reduced norm map

$$\text{Nrd} = \text{Nrd}_A : A \longrightarrow \zeta(A) = F_1 \oplus \cdots \oplus F_t$$

is defined componentwise and extends to matrix rings over A in the obvious way (see [CR81, §7D]). If every D_i is in fact a field, then the reduced norm of $x = (x_i)_i \in A$ is indeed given by $\text{Nrd}(x) = (\det(x_i))_i$. In general, one can always choose a (finite) field extension E of F such that $A_E := E \otimes_F A$ is of this form. Then one puts $\text{Nrd}_A(x) := \text{Nrd}_{A_E}(1 \otimes x)$ which actually belongs to $\zeta(A)$ and is independent of the choice of E .

Now let Λ be a Fitting order in A over \mathfrak{o} . By [Rei03, Corollary 10.4] we may choose a maximal \mathfrak{o} -order Λ' in A containing Λ . Then Λ' is also a Fitting order over \mathfrak{o} and likewise decomposes into $\Lambda' = \Lambda'_1 \oplus \cdots \oplus \Lambda'_t$, where Λ'_i is a maximal \mathfrak{o} -order in A_i for each i . The reduced norm restricts to a map

$$\text{Nrd} : \Lambda' \longrightarrow \zeta(\Lambda') = \mathfrak{o}_1 \oplus \cdots \oplus \mathfrak{o}_t,$$

where $\mathfrak{o}_i = \zeta(\Lambda'_i)$ denotes the integral closure of \mathfrak{o} in F_i . Unfortunately, it is in general not true that the reduced norm maps Λ into its centre.

Example 2.8. Let p be an odd prime and let D_{2p} be the dihedral group of order $2p$. We may write

$$D_{2p} = \langle \sigma, \tau \mid \sigma^p = \tau^2 = 1, \tau\sigma = \sigma^{-1}\tau \rangle.$$

Then $\Lambda := \mathbb{Z}_p[D_{2p}]$ is a Fitting order in $A := \mathbb{Q}_p[D_{2p}]$ over \mathbb{Z}_p and we wish to compute $\text{Nrd}(\sigma + \tau)$. We put $E := \mathbb{Q}_p(\zeta_p)$, where ζ_p denotes a primitive p -th root of unity, and let $j \in \text{Gal}(E/\mathbb{Q}_p)$ be the unique automorphism of order 2. By [CR81, Example 7.39] we have the Wedderburn decomposition

$$(2.1) \quad A \simeq A_1 \oplus A_2 \oplus A_3,$$

where $A_1 = A_2 = \mathbb{Q}_p$ and A_3 is the twisted group algebra $E \oplus Ey$ with relations $y^2 = 1$ and $ya = j(\alpha)y$, $\alpha \in E$. Moreover, for $\alpha + \beta y \in A_3$ one has

$$\text{Nrd}(\alpha + \beta y) = N_{E^+/\mathbb{Q}_p}(\alpha j(\alpha) - \beta j(\beta)),$$

where E^+ denotes the fixed field of E under the action of j and $N_{E^+/\mathbb{Q}_p} : E^+ \rightarrow \mathbb{Q}_p$ is the field-theoretic norm map. The isomorphism (2.1) maps σ to the triple $(1, 1, \zeta_p)$ and τ to the triple $(1, -1, y)$. As the first factor in this decomposition corresponds to the central idempotent $e_1 := \frac{1}{2p} \sum_{\delta \in D_{2p}} \delta$, we have

$$\text{Nrd}(\sigma + \tau) = 2e_1 = \frac{1}{p} \sum_{\delta \in D_{2p}} \delta \notin \zeta(\Lambda).$$

To overcome this problem we define a $\zeta(\Lambda)$ -submodule of $\zeta(A)$ by

$$\mathcal{I}(\Lambda) := \langle \text{Nrd}(H) \mid H \in M_{b \times b}(\Lambda), b \in \mathbb{N} \rangle_{\zeta(\Lambda)}.$$

Note that this is in fact a commutative \mathfrak{o} -order in $\zeta(A)$ contained in $\zeta(\Lambda')$. We call $\mathcal{I}(\Lambda)$ the *integrality ring* of Λ . This is the smallest ring that contains $\zeta(\Lambda)$ and the image of the reduced norm of all matrices with entries in Λ .

Example 2.9. Let ℓ and p be primes with p odd. Choose a maximal \mathbb{Z}_ℓ -order $\mathfrak{M}_\ell(D_{2p})$ containing $\mathbb{Z}_\ell[D_{2p}]$. Then one has (see [JN13, Example 6] and [JN16, Proposition 6.9])

$$\mathcal{I}(\mathbb{Z}_\ell[D_{2p}]) = \begin{cases} \zeta(\mathfrak{M}_p(D_{2p})) & \text{if } \ell = p \\ \zeta(\mathbb{Z}_\ell[D_{2p}]) & \text{if } \ell \neq p. \end{cases}$$

Note that the case $\ell \neq p$ follows from Proposition 3.8 below. For the case $\ell = p$ one has to compute the reduced norms of several group ring elements as in Example 2.8 and then show that these generate $\zeta(\mathfrak{M}_p(D_{2p}))$ as a \mathbb{Z}_p -module.

Remark 2.10. The integrality ring appears in many conjectures on the integrality of so-called Stickelberger elements. These elements lie in the centre of the rational group ring and are constructed via integer values of Artin L -functions attached to the irreducible characters of G , where G is the Galois group of a finite Galois extension of number fields. We refer the reader to [Nic] for a survey of conjectures and results in this context.

2.3. Noncommutative Fitting invariants. Let Λ be a Fitting order over the Fitting domain \mathfrak{o} . Let M be a Λ -module with finite presentation

$$\Lambda^a \xrightarrow{h} \Lambda^b \longrightarrow M \longrightarrow 0.$$

As before we let $S_b(h)$ be the set of all $b \times b$ submatrices of h . Since the reduced norm is a generalisation of the determinant with values in $\mathcal{I}(\Lambda)$, it is now natural to make the following definition.

$$\text{Fitt}_\Lambda(h) := \begin{cases} 0 & \text{if } a < b \\ \langle \text{Nrd}(H) \mid H \in S_b(h) \rangle_{\mathcal{I}(\Lambda)} & \text{if } a \geq b. \end{cases}$$

Unfortunately, this definition depends on h .

Example 2.11. Consider the Fitting order $\Lambda = M_{2 \times 2}(\mathbb{Z}_3)$ over \mathbb{Z}_3 and the trivial Λ -module $M = 0$. We have $\mathcal{I}(\Lambda) = \zeta(\Lambda) = \mathbb{Z}_3$. The identity map $\text{id} : \Lambda \rightarrow \Lambda$ is certainly a finite presentation of M and we have $\text{Fitt}_\Lambda(\text{id}) = \langle \text{Nrd}(\text{id}) \rangle_{\mathbb{Z}_3} = \mathbb{Z}_3$. However, the map

$$\begin{aligned} h : \Lambda e_1 \oplus \Lambda e_2 &\longrightarrow \Lambda \\ e_1 &\mapsto \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \\ e_2 &\mapsto \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \end{aligned}$$

is also a finite presentation of M and we have $\text{Fitt}_\Lambda(h) = \langle 15, 24 \rangle_{\mathbb{Z}_3} = 3\mathbb{Z}_3$.

Remark 2.12. The ring Λ in Example 2.11 is a matrix ring over a commutative ring. In this case one can remedy the dependence on h via Morita equivalence. We will explain this approach in the appendix.

In order to examine the dependence on h we try to adapt the proof of Theorem 1.1 in the commutative case. We still may view a finite presentation of M as a truncated free resolution of M . As Λ is a semiperfect ring (see [Lam01, Example 23.3]), every finitely generated module M has a projective cover (see [CR81, Theorem 6.23] or [Lam01, Theorem 24.16]): there is a finitely generated projective module P_0 (unique up to isomorphism) and a surjective map $\pi : P_0 \rightarrow M$ such that no proper submodule of P_0 is mapped onto M by π . If every projective Λ -module is free, then $P_0 = \Lambda^{b'}$ with $b' \in \mathbb{N}$ minimal such that there is a surjection $\Lambda^{b'} \rightarrow M$. Let P_1 be a projective cover of the kernel of π . We obtain an exact sequence

$$P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

which we may view as the truncation of a ‘minimal projective resolution’ \mathcal{P}_M of M . This is the correct analogue of a minimal free resolution of a module over a commutative local

ring: Any free (in fact any projective) resolution of M is isomorphic to the direct sum of \mathcal{P}_M and a trivial complex [Nic10, Proposition 2.1]. Now let

$$\Lambda^{a'} \xrightarrow{h'} \Lambda^{b'} \longrightarrow M \longrightarrow 0$$

be a second finite presentation of M . By similar arguments as in the commutative case one may assume that $a = a'$, $b = b'$ and that there are matrices $X \in \mathrm{GL}_a(\Lambda)$ and $Y \in \mathrm{GL}_b(\Lambda)$ such that

$$h \circ X = Y \circ h'.$$

As $\mathrm{Nrd}(Y)$ belongs to $\mathcal{I}(\Lambda)^\times$, we may assume in addition that $Y = 1$. In contrast to the determinant, the reduced norm is not a multilinear map so that we cannot assume that $X = 1$. However, assuming $h \circ X = h'$ as we may, we can construct a new finite presentation of M , namely

$$\Lambda^a \oplus \Lambda^a \xrightarrow{(h|h')} \Lambda^b \longrightarrow M \longrightarrow 0.$$

Now $\mathrm{Fitt}_\Lambda((h|h'))$ contains both $\mathrm{Fitt}_\Lambda(h)$ and $\mathrm{Fitt}_\Lambda(h')$. As $\mathcal{I}(\Lambda)$ is a noetherian ring, we have shown the following (see also [Nic10, Theorem 3.2 and Definition 3.3] and [JN13, §3.5]).

Theorem 2.13. *Let Λ be a Fitting order and let M be a finitely generated Λ -module. Then there is a finite presentation h of M such that $\mathrm{Fitt}_\Lambda(h)$ contains $\mathrm{Fitt}_\Lambda(h')$ for every other choice h' of finite presentation of M .*

Definition 2.14. Using the notation of Theorem 2.13, we put

$$\mathrm{Fitt}_\Lambda^{\max}(M) := \mathrm{Fitt}_\Lambda(h)$$

and call this the *maximal Fitting invariant* of M over Λ .

Remark 2.15. For an axiomatic approach to noncommutative Fitting invariants we refer the reader to [Kat18]. A natural notion of ‘higher noncommutative Fitting invariants’ has recently been considered by Burns and Sano [BS17].

Remark 2.16. In order to define noncommutative Fitting invariants it suffices to assume that \mathfrak{o} is a commutative noetherian complete local domain. In fact, the integral closure of \mathfrak{o} in its field of quotients is finitely generated as an \mathfrak{o} -module by [HS06, Theorem 4.3.4] in this case, and noncommutative Fitting invariants have been defined in this greater generality in [Nic10].

One can prove the analogues of Lemma 1.8(i) and (iii) without any significant changes.

Lemma 2.17. *Let Λ be a Fitting order and let M_1, M_2, M_3 be finitely generated Λ -modules.*

- (i) *If $\pi : M_1 \twoheadrightarrow M_2$ is an epimorphism, then $\mathrm{Fitt}_\Lambda^{\max}(M_1) \subseteq \mathrm{Fitt}_\Lambda^{\max}(M_2)$.*
- (ii) *If $M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ is an exact sequence, then*

$$\mathrm{Fitt}_\Lambda^{\max}(M_1) \cdot \mathrm{Fitt}_\Lambda^{\max}(M_3) \subseteq \mathrm{Fitt}_\Lambda^{\max}(M_2).$$

However, the proof of Lemma 1.8(ii) only gives an inclusion

$$\mathrm{Fitt}_\Lambda^{\max}(M_1) \cdot \mathrm{Fitt}_\Lambda^{\max}(M_3) \subseteq \mathrm{Fitt}_\Lambda^{\max}(M_1 \oplus M_3)$$

which is a special case of Lemma 2.17(ii). We will treat the question of whether this inclusion is actually an equality in §5 below.

It is hard to decide in general whether a given presentation gives a maximal Fitting invariant. In this direction we have the following result.

Proposition 2.18. *Let Λ be a Fitting order and let M be a finitely generated Λ -module. If M admits a quadratic presentation h , i.e. a finite presentation of the form*

$$\Lambda^a \xrightarrow{h} \Lambda^a \longrightarrow M \longrightarrow 0,$$

then $\text{Fitt}_\Lambda^{\max}(M) = \text{Fitt}_\Lambda(h)$.

Proof. This follows from [Nic11b, Proposition 1.1 (4)]. □

Remark 2.19. We briefly discuss the relation of noncommutative Fitting invariants to algebraic K -theory. This will not be used in the following. For background on algebraic K -theory we refer the reader to [CR87] and [Swa68].

Suppose that M admits a quadratic presentation h which is injective. Then M is torsion as an \mathfrak{o} -module and therefore defines a class $[M]$ in the relative algebraic K -group $K_0(\Lambda, A)$ associated to the ring homomorphism $\Lambda \hookrightarrow A$. Moreover, the matrix h belongs to $\text{GL}_a(A)$ and so defines a class $[h]$ in $K_1(A)$ such that $\partial([h]) = [M]$, where $\partial : K_1(A) \rightarrow K_0(\Lambda, A)$ denotes the connecting homomorphism of relative algebraic K -theory. The reduced norm induces a group homomorphism $\text{Nrd} : K_1(A) \rightarrow \zeta(A)^\times$ such that $\text{Nrd}([h]) = \text{Nrd}(h)$. Now suppose that $x \in K_1(A)$ is a second pre-image of $[M]$. Then $[h]x^{-1}$ lies in the image of $K_1(\Lambda)$ and therefore $\text{Nrd}(x) = \text{Nrd}([h]) \cdot \text{Nrd}(y)$ for some $y \in K_1(\Lambda)$. As $\text{Nrd}(y) \in \mathcal{I}(\Lambda)^\times$, the $\mathcal{I}(\Lambda)$ -ideals generated by $\text{Nrd}([h])$ and $\text{Nrd}(x)$ coincide. In other words, for any $x \in K_1(A)$ such that $\partial(x) = [M]$ one has

$$\text{Fitt}_\Lambda^{\max}(M) = \langle \text{Nrd}(x) \rangle_{\mathcal{I}(\Lambda)}.$$

Now suppose that $\text{Fitt}_\Lambda^{\max}(M)$ is generated by some $\xi \in \zeta(A)^\times$. Then $\xi \text{Nrd}(x)^{-1}$ belongs to $\mathcal{I}(\Lambda)^\times$, but we cannot conclude in general that $\xi \text{Nrd}(x)^{-1}$ lies in $\text{Nrd}(K_1(\Lambda))$. The more involved notion of $\text{Nrd}(\Lambda)$ -equivalence classes in [Nic10] is designed in such a way that this conclusion works.

3. FITTING INVARIANTS AND ANNIHILATION

3.1. Generalised adjoint matrices. If R is a commutative ring and M is a finitely presented R -module, we know by Theorem 1.5 that $\text{Fitt}_R(M)$ is always contained in the R -annihilator ideal of M . The main ingredient of the proof was the existence of adjoint matrices. We now generalise this concept.

Let Λ be a Fitting order. Choose $n \in \mathbb{N}$ and let $H \in M_{n \times n}(\Lambda)$. Then recalling the notation of §2.2, decompose H into

$$H = \sum_{i=1}^t H_i \in M_{n \times n}(\Lambda') = \bigoplus_{i=1}^t M_{n \times n}(\Lambda'_i).$$

Let $m_i = n_i \cdot s_i \cdot n$, where s_i denotes the Schur index of D_i so that $[D_i : F_i] = s_i^2$. The reduced characteristic polynomial $f_i(X) = \sum_{j=0}^{m_i} \alpha_{ij} X^j$ of H_i has coefficients in \mathfrak{o}_i . Moreover, the constant term α_{i0} is equal to $\text{Nrd}(H_i) \cdot (-1)^{m_i}$. We put

$$H_i^* := (-1)^{m_i+1} \cdot \sum_{j=1}^{m_i} \alpha_{ij} H_i^{j-1}, \quad H^* := \sum_{i=1}^t H_i^*.$$

We call H^* the *generalised adjoint matrix* of H .

Lemma 3.1. *We have $H^* \in M_{n \times n}(\Lambda')$ and $H^*H = HH^* = \text{Nrd}(H) \cdot \mathbf{1}_n$.*

Proof. The first assertion is clear by the above considerations. Since $f_i(H_i) = 0$, we find that

$$H_i^* \cdot H_i = H_i \cdot H_i^* = (-1)^{m_i+1}(-\alpha_{i0}) = \text{Nrd}(H_i),$$

as desired. \square

Remark 3.2. Note that the above definition of H^* differs slightly from the definition in [Nic10, §4]. Here we follow the treatment in [JN13, §3.6].

Remark 3.3. Let E/F be a separable field extension such that $A_E := E \otimes_F A$ splits. We may view H as an element of $M_{n \times n}(A_E)$ which is a finite sum of matrix rings over E . Then $H^* \in M_{n \times n}(A_E)$ is just the sum of the adjoint matrices in each component. As such it might have been more natural to call H^* the ‘reduced adjoint matrix’ of H .

Example 3.4. Let $0 \in M_{n \times n}(R)$ where R is a commutative ring. Then for the adjoint matrix 0^* we have $0^* = 1$ if $n = 1$ and $0^* = 0$ if $n > 1$. Let p be a prime and let G be a finite group. Denote the commutator subgroup of G by G' and let E/\mathbb{Q}_p be a splitting field for $\mathbb{Q}_p[G]$. Then Wedderburn’s theorem for the algebra $E[G]$ implies that for $0 \in M_{1 \times 1}(\mathbb{Z}_p[G])$ we have

$$0^* = \frac{1}{|G'|} \sum_{g \in G'} g.$$

Example 3.5. Let Λ be a Fitting order and let $H \in M_{n \times n}(\Lambda)$. Then for every positive integer m one has

$$\begin{pmatrix} H & 0 \\ 0 & \mathbf{1}_m \end{pmatrix}^* = \begin{pmatrix} H^* & 0 \\ 0 & \text{Nrd}(H)\mathbf{1}_m \end{pmatrix}.$$

In view of Remark 3.3, this follows from the respective statement for adjoint matrices over commutative rings (for a more detailed proof see [Wat18, Theorem 1.7.8(iii)]).

3.2. Denominator ideals. We define

$$\mathcal{H}(\Lambda) := \{x \in \zeta(\Lambda) \mid xH^* \in M_{b \times b}(\Lambda) \forall H \in M_{b \times b}(\Lambda) \forall b \in \mathbb{N}\}$$

and call $\mathcal{H}(\Lambda)$ the *denominator ideal* of Λ . We claim that

$$(3.1) \quad \mathcal{H}(\Lambda) \cdot \mathcal{I}(\Lambda) = \mathcal{H}(\Lambda) \subseteq \zeta(\Lambda)$$

and so $\mathcal{H}(\Lambda)$ is in fact an ideal in the \mathfrak{o} -order $\mathcal{I}(\Lambda)$. We follow an argument of David Watson [Wat18, Lemma 1.10.9]. Let $x \in \mathcal{H}(\Lambda)$ and $H_1 \in M_{b_1 \times b_1}(\Lambda)$, $H_2 \in M_{b_2 \times b_2}(\Lambda)$ with positive integers b_1 and b_2 . We have to show that $x\text{Nrd}(H_1)H_2^*$ belongs to $M_{b_2 \times b_2}(\Lambda)$. By Example 3.5 we may assume that $b_1 = b_2$. We now compute

$$x\text{Nrd}(H_1)H_2^* = xH_1H_1^*H_2^* = H_1x(H_2H_1)^* \in M_{b_2 \times b_2}(\Lambda).$$

Remark 3.6. The denominator ideal $\mathcal{H}(\Lambda)$ measures the failure of the generalised adjoint matrices to have entries in Λ .

Remark 3.7. Let Λ' be a maximal order containing Λ . The central conductor of Λ' over Λ is defined to be $\mathcal{F}(\Lambda) := \{x \in \zeta(\Lambda') \mid x\Lambda' \subseteq \Lambda\}$. It is clear from Lemma 3.1 that we always have $\mathcal{F}(\Lambda) \subseteq \mathcal{H}(\Lambda)$. Note that in particular we have $\mathcal{H}(\Lambda') = \zeta(\Lambda')$ for every maximal Fitting order Λ' .

We now consider the case of p -adic group rings in more detail. If p is a prime and G is a finite group, we set

$$\mathcal{I}_p(G) := \mathcal{I}(\mathbb{Z}_p[G]), \quad \mathcal{H}_p(G) := \mathcal{H}(\mathbb{Z}_p[G]).$$

Proposition 3.8. *Let p be prime and G be a finite group. Then $\mathcal{H}_p(G) = \zeta(\mathbb{Z}_p[G])$ if and only if p does not divide the order of the commutator subgroup of G . Moreover, in this case we have $\mathcal{I}_p(G) = \zeta(\mathbb{Z}_p[G])$.*

Proof. The first claim is a special case of [JN13, Proposition 4.4]. Note that Example 3.4 shows that $\mathcal{H}_p(G) = \zeta(\mathbb{Z}_p[G])$ is only possible if p does not divide the order of the commutator subgroup of G . The second claim follows easily from (3.1). \square

Let $\overline{\mathbb{Q}}_p$ be a separable closure of \mathbb{Q}_p . For an irreducible character $\chi : G \rightarrow \overline{\mathbb{Q}}_p$ we put $\mathbb{Q}_p(\chi) := \mathbb{Q}_p(\chi(g) \mid g \in G)$. In the case of p -adic group rings the central conductor is explicitly given by Jacobinski's formula [Jac66] (see [CR81, Theorem 27.13])

$$(3.2) \quad \mathcal{F}_p(G) := \mathcal{F}(\mathbb{Z}_p[G]) = \bigoplus_x \frac{|G|}{\chi(1)} \mathcal{D}^{-1}(\mathbb{Q}_p(\chi)/\mathbb{Q}_p),$$

where $\mathcal{D}^{-1}(\mathbb{Q}_p(\chi)/\mathbb{Q}_p)$ denotes the inverse different of the extension $\mathbb{Q}_p(\chi)$ over \mathbb{Q}_p and the sum runs over all irreducible characters of G modulo the natural action of the absolute Galois group of \mathbb{Q}_p on the irreducible characters of G .

Example 3.9. Let p and ℓ be primes with p odd. We consider the group ring $\mathbb{Z}_\ell[D_{2p}]$, where D_{2p} denotes the dihedral group of order $2p$. In the case $p = 3$, one has $D_6 \simeq S_3$, the symmetric group on three letters. Then we have

$$\mathcal{H}_\ell(D_{2p}) = \begin{cases} \zeta(\mathbb{Z}_\ell[D_{2p}]) & \text{if } p \neq \ell \\ \mathcal{F}_p(D_{2p}) & \text{if } p = \ell. \end{cases}$$

In fact, the result follows from Proposition 3.8 if $p \neq \ell$. In the case $p = \ell$, the result is established in [JN13, Example 6]. The corresponding integrality rings have already been determined in Example 2.9.

Example 3.10. Let p be a prime and let $q = \ell^n$ be a prime power. We consider the group $\text{Aff}(q) = \mathbb{F}_q \rtimes \mathbb{F}_q^\times$ of affine transformations on \mathbb{F}_q , the finite field with q elements. Let $\mathfrak{M}_p(\text{Aff}(q))$ be a maximal \mathbb{Z}_p -order such that $\mathbb{Z}_p[\text{Aff}(q)] \subseteq \mathfrak{M}_p(\text{Aff}(q)) \subseteq \mathbb{Q}_p[\text{Aff}(q)]$. Then by [JN16, Proposition 6.7] we have

$$\mathcal{H}_p(\text{Aff}(q)) = \begin{cases} \zeta(\mathbb{Z}_p[\text{Aff}(q)]) & \text{if } p \neq \ell \\ \mathcal{F}_p(\text{Aff}(q)) & \text{if } p = \ell \neq 2; \end{cases}$$

$$\mathcal{I}_p(\text{Aff}(q)) = \begin{cases} \zeta(\mathbb{Z}_p[\text{Aff}(q)]) & \text{if } p \neq \ell \\ \zeta(\mathfrak{M}_p(\text{Aff}(q))) & \text{if } p = \ell \neq 2. \end{cases}$$

If $p = \ell = 2$, then we have containments

$$2\mathcal{H}_2(\text{Aff}(q)) \subseteq \mathcal{F}_2(\text{Aff}(q)) \subseteq \mathcal{H}_2(\text{Aff}(q)),$$

$$2\zeta(\mathfrak{M}_2(\text{Aff}(q))) \subseteq \mathcal{I}_2(\text{Aff}(q)) \subseteq \zeta(\mathfrak{M}_2(\text{Aff}(q))).$$

Note that the commutator subgroup of $\text{Aff}(q)$ is \mathbb{F}_q so that the case $p \neq \ell$ again follows from Proposition 3.8. An exact formula for the denominator ideal including the case $p = \ell = 2$ has been determined by David Watson [Wat18, Example 3.6.6]

Example 3.11. Let S_4 be the symmetric group on 4 letters. If p is an odd prime, then $\mathcal{I}_p(S_4) = \zeta(\mathfrak{M}_p(S_4))$ and $\mathcal{H}_p(S_4) = \mathcal{F}_p(S_4)$. However, if $p = 2$ we have

$$\mathcal{F}_2(S_4) \subsetneq \mathcal{H}_2(S_4) \subsetneq \zeta(\mathbb{Z}_2[S_4]);$$

$$\zeta(\mathbb{Z}_2[S_4]) \subsetneq \mathcal{I}_2(S_4) \subsetneq \zeta(\mathfrak{M}_2(S_4)).$$

This follows from [JN16, Proposition 6.8].

Remark 3.12. Even in the case of p -adic group rings, a general formula for denominator ideals is still not available, though it would be of significant interest for arithmetic applications. In particular, we seek good lower bounds. This question is extensively studied in the PhD thesis of David Watson [Wat18]. In particular, he determines the denominator ideal $\mathcal{H}_p(G)$ for any (non-abelian) group G of order p^3 .

Now let Λ be an arbitrary Fitting order and let Λ' be a maximal order containing Λ . We define a variant of the central conductor by

$$\mathcal{F}_\zeta(\Lambda) := \{x \in \zeta(\Lambda') \mid x\zeta(\Lambda') \subseteq \zeta(\Lambda)\}.$$

One clearly has an inclusion $\mathcal{F}(\Lambda) \subseteq \mathcal{F}_\zeta(\Lambda)$, but this is not an equality in general.

Example 3.13. Let D_{2^a} be the dihedral group of order 2^a , where $a \geq 3$. Then one can show (see [JN13, Example 7]) that

$$[\mathcal{F}_\zeta(\mathbb{Z}_2[D_{2^a}]) : \mathcal{F}(\mathbb{Z}_2[D_{2^a}])] = 2^{a-2}.$$

Remark 3.14. In the case where Λ is a p -adic group ring one has an explicit formula for $\mathcal{F}_\zeta(\Lambda)$; see [JN13, Proposition 6.12].

Since the reduced characteristic polynomials have coefficients in $\zeta(\Lambda')$, one can give the following lower bound [JN13, Proposition 6.3] for $\mathcal{H}(\Lambda)$.

Proposition 3.15. *We have $\mathcal{F}_\zeta(\Lambda) \subseteq \mathcal{H}(\Lambda)$.*

3.3. Fitting invariants and annihilation. Now a proof similar to the commutative case shows the desired annihilation result (see [Nic10, Theorem 4.2] and [JN13, Theorem 3.3]).

Theorem 3.16. *Let Λ be a Fitting order and let M be a finitely generated Λ -module. Then one has an inclusion*

$$\mathcal{H}(\Lambda) \cdot \text{Fitt}_\Lambda^{\max}(M) \subseteq \text{Ann}_{\zeta(\Lambda)}(M).$$

Since $\mathcal{F}(\Lambda)$ is contained in $\mathcal{H}(\Lambda)$, the above inclusion also holds with $\mathcal{H}(\Lambda)$ replaced by $\mathcal{F}(\Lambda)$. However, if one wishes to compute annihilators using $\mathcal{F}(\Lambda)$ then the following result shows that it suffices to compute the Fitting invariant over the maximal order (see [JN13, Corollary 6.5 and Theorem 6.7]).

Proposition 3.17. *Let Λ be a Fitting order and let M be a finitely generated Λ -module. Choose a maximal order Λ' containing Λ . Then*

$$\mathcal{F}(\Lambda) \cdot \text{Fitt}_\Lambda^{\max}(M) \subseteq \mathcal{F}(\Lambda) \cdot \text{Fitt}_{\Lambda'}^{\max}(\Lambda' \otimes_\Lambda M) \subseteq \text{Ann}_{\zeta(\Lambda)}(M).$$

Example 3.18. We generalise Example 1.7. Let p be a prime and let G be a finite group. Let $\Delta_p G$ be the kernel of the natural augmentation map $\text{aug}_p : \mathbb{Z}_p[G] \rightarrow \mathbb{Z}_p$ that sends each $g \in G$ to 1. As $|G|$ belongs to $\mathcal{H}_p(G)$ we have by Theorem 3.16 that

$$|G| \cdot \text{Fitt}_{\mathbb{Z}_p[G]}^{\max}(\Delta_p G) \subseteq \text{Ann}_{\mathbb{Z}_p[G]}(\Delta_p G) = N_G \mathbb{Z}_p,$$

where as before $N_G := \sum_{g \in G} g$. It follows that

$$\text{Fitt}_{\mathbb{Z}_p[G]}^{\max}(\Delta_p G) = m \frac{1}{|G|} N_G \mathbb{Z}_p$$

for some $m \in \mathbb{Z}_p$. Let h be a finite presentation of $\Delta_p G$ such that $\text{Fitt}_{\mathbb{Z}_p[G]}(h) = \text{Fitt}_{\mathbb{Z}_p[G]}^{\max}(\Delta_p G)$. Let $\text{aug}_p(h)$ be the matrix obtained from h by applying aug_p to each of the entries of h . Then $\text{aug}_p(h)$ is a finite presentation of the \mathbb{Z}_p -module

$$\mathbb{Z}_p \otimes_{\mathbb{Z}_p[G]} \Delta_p G \simeq \Delta_p G / (\Delta_p G)^2 \simeq \mathbb{Z}_p \otimes_{\mathbb{Z}} G / G',$$

where G' denotes the commutator subgroup of G . It follows from Example 1.6 that

$$m\mathbb{Z}_p = \text{Fitt}_{\mathbb{Z}_p}(\text{aug}_p(h)) = \text{Fitt}_{\mathbb{Z}_p}(\mathbb{Z}_p \otimes_{\mathbb{Z}_p[G]} \Delta_p G) = |G/G'| \mathbb{Z}_p.$$

This implies that we may choose $m = |G/G'|$ and thus

$$\text{Fitt}_{\mathbb{Z}_p[G]}^{\max}(\Delta_p G) = \frac{1}{|G'|} N_G \mathbb{Z}_p.$$

As $N_{G'} := \sum_{g \in G'} g$ belongs to $\mathcal{H}_p(G)$ by [JN13, Corollary 6.14] we find that

$$\mathcal{H}_p(G) \cdot \text{Fitt}_{\mathbb{Z}_p[G]}^{\max}(\Delta_p G) = N_G \mathbb{Z}_p = \text{Ann}_{\mathbb{Z}_p[G]}(\Delta_p G).$$

Let $\mathfrak{M}_p(G)$ be a maximal order containing $\mathbb{Z}_p[G]$. One can likewise show that

$$\text{Fitt}_{\mathfrak{M}_p(G)}^{\max}(\mathfrak{M}_p(G) \otimes_{\mathbb{Z}_p[G]} \Delta_p G) = \frac{1}{|G'|} N_G \mathbb{Z}_p.$$

Then Proposition 3.17 implies the weaker result

$$\mathcal{F}(\mathbb{Z}_p[G]) \cdot \text{Fitt}_{\mathfrak{M}_p(G)}^{\max}(\mathfrak{M}_p(G) \otimes_{\mathbb{Z}_p[G]} \Delta_p G) = \frac{|G|}{|G'|} N_G \mathbb{Z}_p \subseteq \text{Ann}_{\mathbb{Z}_p[G]}(\Delta_p G).$$

4. p -ADIC GROUP RINGS

In this section we fix a prime p and a finite group G . The p -adic group ring $\mathbb{Z}_p[G]$ is a Fitting order over \mathbb{Z}_p which is of particular interest in number theory. We put $\Lambda := \mathbb{Z}_p[G]$ and $A := \mathbb{Q}_p[G]$.

For any Λ -module M we write M^\vee for its Pontryagin dual $\text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ and M^* for the linear dual $\text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$, each endowed with the natural contragredient action of G . We denote by $\sharp : A \rightarrow A$ the anti-involution which maps each $g \in G$ to its inverse. If $h \in M_{a \times b}(A)$ is a matrix we let h^\sharp be the matrix obtained from h by applying \sharp to each of its entries. Moreover, we let $h^T \in M_{b \times a}(A)$ be the transpose of h . We note that there is an isomorphism $\Lambda^* \simeq \Lambda$, $f \mapsto \sum_{g \in G} f(g)g$. Under this identification the \mathbb{Z}_p -dual of a map $h \in M_{a \times b}(\Lambda)$ identifies with $h^{T, \sharp} \in M_{b \times a}(\Lambda)$.

Now let C be a finite Λ -module of projective dimension at most 1. Choose $n \in \mathbb{N}$ and a surjective map $\Lambda^n \twoheadrightarrow C$ with kernel P . Note that P is projective. As C is finite, we have an isomorphism $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} P \simeq A^n$ of A -modules. Now Swan's theorem [CR81, Theorem (32.1)] implies that in fact $P \simeq \Lambda^n$. In particular, we find that C has a quadratic presentation

$$(4.1) \quad 0 \longrightarrow \Lambda^n \xrightarrow{q} \Lambda^n \longrightarrow C \longrightarrow 0.$$

Moreover, the maximal Fitting invariant $\text{Fitt}_\Lambda^{\max}(C)$ is generated by $\text{Nrd}(q) \in \zeta(A)^\times$ by Proposition 2.18.

We also note that a Λ -module is of projective dimension at most 1 if and only if it is a cohomologically trivial Λ -module by [AW67, Theorem 9]. The following result is very useful in computing Fitting invariants over p -adic group rings.

Proposition 4.1. *Let $\Lambda := \mathbb{Z}_p[G]$ where p is a prime and G is a finite group.*

- (i) *Let C be a finite Λ -module of projective dimension at most 1. Let $c \in \zeta(A)^\times$ be a generator of $\text{Fitt}_\Lambda^{\max}(C)$. Then the Pontryagin dual C^\vee is also a finite Λ -module of projective dimension at most 1 and $\text{Fitt}_\Lambda^{\max}(C^\vee)$ is generated by c^\sharp .*

(ii) Suppose we are given an exact sequence of finite Λ -modules

$$0 \longrightarrow M \longrightarrow C \longrightarrow C' \longrightarrow M' \longrightarrow 0,$$

where C and C' are of projective dimension at most 1. Then we have an equality

$$\mathrm{Fitt}_{\Lambda}^{\max}(M^{\vee})^{\sharp} \cdot \mathrm{Fitt}_{\Lambda}^{\max}(C') = \mathrm{Fitt}_{\Lambda}^{\max}(M') \cdot \mathrm{Fitt}_{\Lambda}^{\max}(C).$$

Proof. This follows from [Nic10, Proposition 5.3]. Here we will give a new proof of (i) which is much shorter and easier than the original proof. The argument is inspired by [BG03, Lemma 6] and recent work of Kataoka [Kat18, §4].

As C is finite, we have $\mathrm{Hom}_{\mathbb{Z}_p}(C, \mathbb{Z}_p) = \mathrm{Hom}_{\mathbb{Z}_p}(C, \mathbb{Q}_p) = 0$. As \mathbb{Q}_p is an injective \mathbb{Z}_p -module, we have $\mathrm{Ext}_{\mathbb{Z}_p}^1(C, \mathbb{Q}_p) = 0$ and thus the short exact sequence $\mathbb{Z}_p \hookrightarrow \mathbb{Q}_p \twoheadrightarrow \mathbb{Q}_p/\mathbb{Z}_p$ induces an isomorphism

$$C^{\vee} \simeq \mathrm{Ext}_{\mathbb{Z}_p}^1(C, \mathbb{Z}_p).$$

Now choose a quadratic presentation q of C as in (4.1). We may assume that $c = \mathrm{Nrd}(q)$. Note that $\mathrm{Ext}_{\mathbb{Z}_p}^1(\Lambda, \mathbb{Z}_p)$ vanishes, since Λ is a projective \mathbb{Z}_p -module. We apply \mathbb{Z}_p -duals to (4.1) and obtain an exact sequence

$$0 \longrightarrow \Lambda^n \xrightarrow{q^{T, \sharp}} \Lambda^n \longrightarrow \mathrm{Ext}_{\mathbb{Z}_p}^1(C, \mathbb{Z}_p) \longrightarrow 0.$$

As $\mathrm{Nrd}(q^{T, \sharp}) = c^{\sharp}$ we are done. \square

Remark 4.2. Note that exact sequences of the type considered in Proposition 4.1 naturally occur in the context of the equivariant Tamagawa number conjecture as formulated by Burns and Flach [BF01]. This conjecture refines and generalises a very wide range of well known results and conjectures relating special values of L -functions to certain natural arithmetic invariants. It thereby vastly generalises the analytic class number formula for number fields and the Birch and Swinnerton-Dyer conjecture for elliptic curves (see [Fla04] for a survey).

Remark 4.3. There is also an analogue of Proposition 4.1 for Iwasawa modules [Nic10, Proposition 6.3] and even for more general Fitting orders [Kat18, §4]. This has applications in the context of main conjectures of equivariant Iwasawa theory.

5. ADDITIVITY OF FITTING INVARIANTS

Let Λ be a Fitting order over the Fitting domain \mathfrak{o} . Let M and N be two finitely generated Λ -modules. As already observed in Lemma 2.17(ii), one always has an inclusion

$$\mathrm{Fitt}_{\Lambda}^{\max}(M) \cdot \mathrm{Fitt}_{\Lambda}^{\max}(N) \subseteq \mathrm{Fitt}_{\Lambda}^{\max}(M \oplus N).$$

Definition 5.1. The Fitting order Λ is called *Fitting-additive* if

$$\mathrm{Fitt}_{\Lambda}^{\max}(M) \cdot \mathrm{Fitt}_{\Lambda}^{\max}(N) = \mathrm{Fitt}_{\Lambda}^{\max}(M \oplus N)$$

for all finitely generated Λ -modules M and N .

The following observation is clear by Lemma 1.8(ii).

Proposition 5.2. *Every commutative Fitting order Λ is Fitting-additive.*

As reduced norms are defined componentwise, the following is also immediate.

Lemma 5.3. *Let Λ_1 and Λ_2 be Fitting orders over the Fitting domain \mathfrak{o} . Then $\Lambda_1 \oplus \Lambda_2$ is Fitting-additive if and only if both Λ_1 and Λ_2 are Fitting-additive.*

We record some cases where it is known that Λ is Fitting-additive.

Theorem 5.4. *The Fitting order Λ is Fitting-additive in each of the following cases.*

- (i) Λ is a direct product of matrix rings over commutative rings.
- (ii) Λ is a maximal order and \mathfrak{o} is a complete discrete valuation ring.

Proof. This follows from [JN13, Theorem 4.6(ii)]. We will reprove part (i) in the appendix (see Remark A.3 and Lemma A.10(ii)). \square

Corollary 5.5. *Let p be a prime and let G be a finite group. Suppose that p does not divide the order of the commutator subgroup of G . Then the p -adic group ring $\mathbb{Z}_p[G]$ is Fitting-additive.*

Proof. It follows from [DJ83, Corollary, p. 390] that $\mathbb{Z}_p[G]$ is a direct product of matrix rings over commutative rings in this case. Thus the result follows from Theorem 5.4(i). \square

Corollary 5.6. *Let G be a profinite group containing a finite normal subgroup H such that $G/H \simeq \mathbb{Z}_p$ for some prime p . Suppose that p does not divide the order of the commutator subgroup of G (which is finite). Then the Iwasawa algebra $\mathbb{Z}_p[[G]]$ is Fitting-additive.*

Proof. The Iwasawa algebra $\mathbb{Z}_p[[G]]$ is again a direct product of matrix rings over commutative rings in this case by [JN13, Proposition 4.5]. \square

We now give an example of a Fitting order Λ which is *not* Fitting-additive.

Example 5.7. Let p be a prime and consider the \mathbb{Z}_p -order

$$\Lambda := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2 \times 2}(\mathbb{Z}_p) \mid b \equiv 0 \pmod{p} \right\}.$$

This is a Fitting order over \mathbb{Z}_p and one has $\mathcal{H}(\Lambda) = \mathcal{I}(\Lambda) = \zeta(\Lambda) = \mathbb{Z}_p$. We let M and N be Λ -modules which as sets are equal to $\mathbb{Z}_p/p\mathbb{Z}_p$ and upon which Λ acts as follows.

Let $\lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Lambda$. For every $x \in \mathbb{Z}_p$ we write \bar{x} for its image in $\mathbb{Z}_p/p\mathbb{Z}_p$. Then $\lambda \cdot m := \bar{a}m$ and $\lambda \cdot n := \bar{d}n$ for $m \in M$ and $n \in N$. Using $\bar{b} = 0$ it is easily checked that this defines left Λ -module structures on M and N , respectively. There is a short exact sequence

$$0 \longrightarrow \Lambda \xrightarrow{h} \Lambda \longrightarrow M \oplus N \longrightarrow 0,$$

where h is right multiplication by $\begin{pmatrix} 0 & p \\ 1 & 0 \end{pmatrix} \in \Lambda$. As h is a quadratic presentation, we have that

$$\text{Fitt}_{\Lambda}^{\max}(M \oplus N) = \text{Nrd}(h) \cdot \mathbb{Z}_p = p\mathbb{Z}_p$$

by Proposition 2.18. As $M \oplus N$ surjects onto M , the ideal generated by p is contained in $\text{Fitt}_{\Lambda}^{\max}(M)$ by Lemma 2.17(i). However, the maximal Fitting invariant $\text{Fitt}_{\Lambda}^{\max}(M)$ annihilates M by Theorem 3.16 and so $\text{Fitt}_{\Lambda}^{\max}(M)$ is properly contained in \mathbb{Z}_p as $M \neq 0$. It follows that

$$\text{Fitt}_{\Lambda}^{\max}(M) = p\mathbb{Z}_p.$$

Exactly the same reasoning applies for N and therefore

$$\text{Fitt}_{\Lambda}^{\max}(N) = p\mathbb{Z}_p.$$

Altogether we have that

$$\text{Fitt}_{\Lambda}^{\max}(M) \cdot \text{Fitt}_{\Lambda}^{\max}(N) = p^2\mathbb{Z}_p \subsetneq p\mathbb{Z}_p = \text{Fitt}_{\Lambda}^{\max}(M \oplus N)$$

and thus Λ is not Fitting-additive.

The order in Example 5.7 is a hereditary, but non-maximal \mathbb{Z}_p -order. By the classification of hereditary orders over complete discrete valuation rings [CR81, Theorem 26.28] it is clear that similar examples can be constructed for every hereditary, non-maximal order over a complete discrete valuation ring. Taking Theorem 5.4(ii) into account, we have established the following.

Proposition 5.8. *Let Λ be a Fitting order over a complete discrete valuation ring. Suppose that Λ is hereditary. Then Λ is Fitting-additive if and only if it is maximal.*

Remark 5.9. A p -adic group ring $\mathbb{Z}_p[G]$ is hereditary if and only if it is maximal. We give an indirect proof of this fact. Suppose that $\mathbb{Z}_p[G]$ is hereditary. Then $\mathcal{H}_p(G) = \zeta(\mathbb{Z}_p[G])$ by [JN13, Corollary 4.2] (this follows easily from the definitions once one observes that the centre of a hereditary \mathbb{Z}_p -order is itself a maximal \mathbb{Z}_p -order). Now Proposition 3.8 implies that p does not divide the order of the commutator subgroup of G . By Corollary 5.5 the group ring $\mathbb{Z}_p[G]$ is Fitting-additive and so by Proposition 5.8 it is maximal.

Let Λ be a Fitting order over the Fitting domain \mathfrak{o} . In view of Example 5.7 and Proposition 5.8 one may ask whether there are hereditary, non-maximal orders when \mathfrak{o} is not a complete discrete valuation ring. We now show that this is indeed not the case.

Proposition 5.10. *Let Λ be a Fitting order over the Fitting domain \mathfrak{o} . If Λ is hereditary, then \mathfrak{o} is a complete discrete valuation ring.*

Proof. Suppose that Λ is a hereditary Fitting order over \mathfrak{o} in the separable F -algebra A , where as before F denotes the quotient field of \mathfrak{o} . Let $A = A_1 \oplus \cdots \oplus A_t$ be the Wedderburn decomposition of A so that each A_i is a central simple $F_i = \zeta(A_i)$ -algebra. Let \mathfrak{o}_i be the integral closure of \mathfrak{o} in F_i . Then we likewise have a decomposition

$$\Lambda = \Lambda_1 \oplus \cdots \oplus \Lambda_t,$$

where each Λ_i is a hereditary \mathfrak{o}_i -order by [Har63, Proposition 2.2]. Moreover, each \mathfrak{o}_i is in fact a Dedekind domain by [Har63, Theorem 2.6] and thus has Krull dimension 1. The Fitting domain \mathfrak{o} then also has Krull dimension 1 by [Eis95, Proposition 4.15]. However, a Fitting domain of Krull dimension 1 is a complete discrete valuation ring. \square

Remark 5.11. In view of Corollary 5.5 and Remark 5.9 one may ask the following question: Is every p -adic group ring Fitting-additive? Similarly, is the Iwasawa algebra $\mathbb{Z}_p[[G]]$ of a one-dimensional p -adic Lie group G always Fitting-additive? In both cases one knows that

$$\text{Fitt}_\Lambda^{\max}(M) \cdot \text{Fitt}_\Lambda^{\max}(N) = \text{Fitt}_\Lambda^{\max}(M \oplus N)$$

whenever at least one of the two Λ -modules M and N has projective dimension at most 1. This follows from [Kat18, Proposition 2.11].

APPENDIX A. FITTING INVARIANTS AND MORITA EQUIVALENCE BY HENRI JOHNSTON AND ANDREAS NICKEL

A.1. Preliminaries on Morita equivalence. Let Λ and R be any two unitary rings. Let ${}_R\mathfrak{M}$ denote the category of left R -modules, ${}_R\mathfrak{M}_\Lambda$ the category of (R, Λ) -bimodules, and so on. For a (R, Λ) -bimodule M , we let ${}_R M$ and M_Λ denote M considered as a left R -module and a right Λ -module, respectively. The rings Λ and R are said to be Morita equivalent if the categories ${}_R\mathfrak{M}$ and ${}_\Lambda\mathfrak{M}$ are equivalent (this is the case if and only if the categories \mathfrak{M}_R and \mathfrak{M}_Λ are equivalent). For example, if $n \in \mathbb{N}$ and $\Lambda = M_{n \times n}(R)$ then it is well-known that Λ and R are Morita equivalent. We shall recall and use some

basic facts on Morita equivalence, and refer the reader to [CR81, §3D], [Rei03, Chapter 4], [Lam99, Chapter 7] or [AF92, §22] for further details.

A generator P for \mathfrak{M}_R is a right R -module such that every right R -module is an epimorphic image of $\bigoplus_{i \in I} P$ for I sufficiently large, or equivalently, if R_R is a direct summand of P^n for some $n \in \mathbb{N}$. A progenerator for \mathfrak{M}_R is a finitely generated projective generator. A progenerator for ${}_{\Lambda}\mathfrak{M}$ is defined analogously.

A (Λ, R) -progenerator is a (Λ, R) -bimodule P such that P_R is a progenerator for \mathfrak{M}_R and the canonical ring homomorphism $\Lambda \rightarrow \text{End}(P_R)$, $\lambda \mapsto (p \mapsto \lambda p)$ is an isomorphism. In this case, ${}_{\Lambda}P$ is a progenerator for ${}_{\Lambda}\mathfrak{M}$ and the canonical ring homomorphism $R^{\text{op}} \rightarrow \text{End}({}_{\Lambda}P)$, $r \mapsto (p \mapsto pr)$ is an isomorphism. For such a P we let $Q = P^* = \text{Hom}_R(P_R, R_R)$ be the R -linear dual of P . Note that Q is an (R, Λ) -bimodule with left action of R defined by $(rq)p = r(qp)$ and right action of Λ defined by $(q\lambda)p = q(\lambda p)$, where $r \in R$, $q \in Q$, $p \in P$ and $\lambda \in \Lambda$. Moreover, $P^* = Q$ is in fact a (R, Λ) -progenerator.

Now suppose that Λ and R are Morita equivalent. Then by a theorem of Morita (see [AF92, Theorem 22.2]) there exists a (Λ, R) -progenerator P such that the functors

$$\begin{aligned} G : {}_R\mathfrak{M} &\longrightarrow {}_{\Lambda}\mathfrak{M}, & N &\mapsto P \otimes_R N \\ F : {}_{\Lambda}\mathfrak{M} &\longrightarrow {}_R\mathfrak{M}, & M &\mapsto P^* \otimes_{\Lambda} M \end{aligned}$$

are mutually inverse category equivalences. Moreover, the isomorphism classes of category equivalences ${}_R\mathfrak{M} \longrightarrow {}_{\Lambda}\mathfrak{M}$ are in one-one correspondence with the isomorphism classes of (Λ, R) -progenerators (see [Lam99, Theorem 18.28]); of course the same statement holds with Λ and R swapped.

Remark A.1. If R is a ring over which all finitely generated projective modules are free (this is the case when R is a local ring, for example) then Λ is Morita equivalent to R if and only if Λ is isomorphic to $M_{n \times n}(R)$ for some $n \in \mathbb{N}$ (see [Lam99, Corollary 18.36]). More generally, if $R = R_1 \oplus \cdots \oplus R_k$ is a direct product of rings and each R_i is a ring over which all finitely generated projective modules are free, then Λ is Morita equivalent to R if and only if Λ is isomorphic to $M_{n_1 \times n_1}(R_1) \oplus \cdots \oplus M_{n_k \times n_k}(R_k)$ for some $n_1, \dots, n_k \in \mathbb{N}$.

A.2. Rings that are Morita equivalent to commutative rings. We now specialise to the situation where Λ is Morita equivalent to its centre $R := \zeta(\Lambda)$. As Morita equivalent rings have isomorphic centres (see [Lam99, Corollary 18.42]), this assumption is the same as supposing that Λ is Morita equivalent to some commutative ring. We recall the convention that all Λ -modules are assumed to be left modules unless stated otherwise.

Definition A.2. Let M be a finitely presented Λ -module. Then we define the Fitting invariant of M over Λ to be the R -ideal

$$\text{Fitt}_{\Lambda}(M) := \text{Fitt}_R(F(M)) = \text{Fitt}_R(P^* \otimes_{\Lambda} M).$$

Remark A.3. Suppose that Λ is both a Fitting order and Morita equivalent to its centre R . We claim that $\text{Fitt}_{\Lambda}(M) = \text{Fitt}_{\Lambda}^{\text{max}}(M)$ for every finitely generated Λ -module M and so the two notions of Fitting invariant coincide in this setting. To see this, note that since Λ is a Fitting order, [Lam01, Example 23.3 and Theorem 23.11] show that we can write $R = R_1 \oplus \cdots \oplus R_k$ where each R_i is a commutative local ring. Thus by Remark A.1, Λ is isomorphic to $M_{n_1 \times n_1}(R_1) \oplus \cdots \oplus M_{n_k \times n_k}(R_k)$ for some $n_1, \dots, n_k \in \mathbb{N}$. The claim now follows by applying [JN13, Proposition 3.4] to each component.

Remark A.4. Suppose that $\Lambda = M_{n \times n}(R)$ for some $n \in \mathbb{N}$ and some commutative ring R . In this situation, Fitting invariants over Λ are defined in [JN13, §2] using an explicit version of the Morita equivalence of Λ and R , and this definition agrees with Definition

A.2. Moreover, it is trivial to extend the definition of *ibid.* to the situation in which Λ is a direct product of matrix rings over commutative rings; again both definitions agree in this situation. However, as we shall see in Example A.6 below, a ring that is Morita equivalent to its centre need not be isomorphic to a product of matrix rings over commutative rings; thus the results presented here extend those of [JN13, §2].

Remark A.5. Suppose that M is a Λ -module that is finitely presented over both Λ and its centre R . Then of course one can consider $\text{Fitt}_R(M)$, but in general this is a coarser invariant than $\text{Fitt}_\Lambda(M)$. For example, if $\Lambda = M_{n \times n}(R)$ for some $n \in \mathbb{N}$ then $\text{Fitt}_R(M) = \text{Fitt}_\Lambda(M)^n$ by [JN13, Theorem 2.2(viii)].

Example A.6. Let R be a Dedekind domain and let $\text{Cl}(R)$ denote its class group. Suppose that there exists a non-zero ideal \mathfrak{a} of R such that the image of \mathfrak{a} in $\text{Cl}(R)/\text{Cl}(R)^2$ is non-trivial. For example, we can take \mathfrak{a} to be any non-principal ideal of $R = \mathbb{Z}[\sqrt{-5}]$. Let

$$\Lambda = \begin{pmatrix} R & \mathfrak{a} \\ \mathfrak{a}^{-1} & R \end{pmatrix}$$

be the ring of all 2×2 matrices (x_{ij}) where x_{11} ranges over all elements of R , x_{12} ranges over all elements of \mathfrak{a} , etc. Let $P = R \oplus \mathfrak{a}$ and note that this is a progenerator for \mathfrak{M}_R . Then $\Lambda \simeq \text{End}_R(P)$ and so Λ is Morita equivalent to R by [Lam99, Proposition 18.33]. However, [BK00, Exercise 4.2.5] shows that Λ is *not* isomorphic to $M_{2 \times 2}(R)$.

Proposition A.7. *The Fitting invariant $\text{Fitt}_\Lambda(M)$ is well-defined.*

Proof. We first note that $F(M)$ is a finitely presented R -module since the property of being finitely presented is preserved under equivalence of module categories (see the discussion in [Lam99, §18A, p. 481], for instance). Now let G' and F' be another pair of mutually inverse category equivalences of ${}_R\mathfrak{M}$ and ${}_\Lambda\mathfrak{M}$. Then the compositions of functors $F' \circ G$ and $F \circ G'$ are mutually inverse category self-equivalences of ${}_R\mathfrak{M}$. Thus the result follows from Lemma A.8 below. \square

Lemma A.8. *Let $T : {}_R\mathfrak{M} \rightarrow {}_R\mathfrak{M}$ be any self-equivalence of ${}_R\mathfrak{M}$. If M is any R -module then $\text{Ann}_R(M) = \text{Ann}_R(T(M))$. Moreover, $\text{Fitt}_R(M) = \text{Fitt}_R(T(M))$ if we further assume that M is finitely presented.*

Proof. By [Lam99, Corollary 18.29] there exists an (R, R) -progenerator W such that the functor $T' : {}_R\mathfrak{M} \rightarrow {}_R\mathfrak{M}$, $M \mapsto W \otimes_R M$ is naturally isomorphic to T . In particular, $T(M)$ and $T'(M)$ are isomorphic as R -modules. Hence $\text{Ann}_R(T(M)) = \text{Ann}_R(T'(M))$ and, if M is finitely presented, $\text{Fitt}_R(T(M)) = \text{Fitt}_R(T'(M))$. Thus we can and do assume without loss of generality that $T = T'$.

Let $x \in \text{Ann}_R(M)$. Then

$$x \cdot T(M) = x \cdot (W \otimes_R M) = W \otimes_R (x \cdot M) = 0.$$

Hence $\text{Ann}_R(M) \subset \text{Ann}_R(T(M))$. Moreover, there exists a functor $U : {}_R\mathfrak{M} \rightarrow {}_R\mathfrak{M}$ such that $U \circ T$ is naturally isomorphic to the identity functor on ${}_R\mathfrak{M}$ and so the same argument gives $\text{Ann}_R(T(M)) \subset \text{Ann}_R(UT(M)) = \text{Ann}_R(M)$. Thus $\text{Ann}_R(M) = \text{Ann}_R(T(M))$.

For the second claim choose a finite presentation $R^a \xrightarrow{h} R^b \twoheadrightarrow M$. As $\text{Fitt}_R(M)$ is generated by the $b \times b$ minors of h , we can and do assume that $a = b$. Hence we may view h as an element of $M_{b \times b}(R)$. Applying T yields an endomorphism

$$T(h) = 1 \otimes h \in \text{End}_R(W \otimes_R R^b) \simeq \text{End}_R(W^b) \simeq M_{b \times b}(R),$$

where the last isomorphism is induced by $\text{End}_R(W) \simeq R$. Thus we have

$$\det(T(h)) = \det(1 \otimes h) = \det(h).$$

This shows $\text{Fitt}_R(M) \subset \text{Fitt}_R(T(M))$ and we again obtain equality by symmetry. \square

Example A.9. Let R be a Dedekind domain and let \mathfrak{a} be a non-zero (fractional) ideal of R . Then \mathfrak{a} is an (R, R) -progenerator and \mathfrak{a}^* naturally identifies with \mathfrak{a}^{-1} . Moreover, Lemma A.8 implies that $\text{Fitt}_R(M) = \text{Fitt}_R(\mathfrak{a} \otimes_R M)$ for every finitely generated R -module M .

Lemma A.10. *Let M_1, M_2, M_3 be finitely presented Λ -modules.*

- (i) *If $\pi : M_1 \rightarrow M_2$ is an epimorphism, then $\text{Fitt}_\Lambda(M_1) \subseteq \text{Fitt}_\Lambda(M_2)$.*
- (ii) *Fitting invariants over Λ behave well under direct sums:*

$$\text{Fitt}_\Lambda(M_1 \oplus M_3) = \text{Fitt}_\Lambda(M_1) \cdot \text{Fitt}_\Lambda(M_3).$$

- (iii) *If $M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ is an exact sequence, then*

$$\text{Fitt}_\Lambda(M_1) \cdot \text{Fitt}_\Lambda(M_3) \subseteq \text{Fitt}_\Lambda(M_2).$$

Proof. The equivalence of categories $F : {}_\Lambda \mathfrak{M} \rightarrow {}_R \mathfrak{M}$ preserves epimorphisms, direct sums and exact sequences by [AF92, Propositions 21.2, 21.4 and 21.5]. Thus the results follow from the corresponding properties of Fitting ideals over R (see Lemma 1.8). \square

Lemma A.11. *Let M be a finitely presented Λ -module. Then for any homomorphism $R \rightarrow S$ of commutative rings, $S \otimes_R M$ is a finitely presented $S \otimes_R \Lambda$ -module and*

$$\text{Fitt}_{S \otimes_R \Lambda}(S \otimes_R M) = S \otimes_R \text{Fitt}_\Lambda(M).$$

Proof. The first claim follows by applying the right exact functor $S \otimes_R -$ to a finite presentation of M . For the second claim, observe that since P^* is a (R, Λ) -progenerator, it is straightforward to check from the definitions that $S \otimes_R P^*$ is a $(S, S \otimes_R \Lambda)$ -progenerator. Thus $(S \otimes_R P^*) \otimes_{S \otimes_R \Lambda} -$ induces an equivalence of categories ${}_{S \otimes_R \Lambda} \mathfrak{M} \rightarrow {}_S \mathfrak{M}$ and so Definition A.2 and Remark 1.3 give

$$\begin{aligned} \text{Fitt}_{S \otimes_R \Lambda}(S \otimes_R M) &= \text{Fitt}_S((S \otimes_R P^*) \otimes_{S \otimes_R \Lambda} (S \otimes_R M)) \\ &= \text{Fitt}_S(S \otimes_R (P^* \otimes_\Lambda M)) \\ &= S \otimes_R \text{Fitt}_R(P^* \otimes_\Lambda M) \\ &= S \otimes_R \text{Fitt}_\Lambda(M). \end{aligned} \quad \square$$

Proposition A.12. *We have $\text{Fitt}_\Lambda(M) \subset \text{Ann}_R(M)$.*

Proof. Let $x \in \text{Fitt}_\Lambda(M) = \text{Fitt}_R(F(M))$. Then x annihilates $F(M) = P^* \otimes_\Lambda M$ by the corresponding property of Fitting ideals over the commutative ring R . Hence x also annihilates $F(M)^k$ for every $k \in \mathbb{N}$. As P^* is in particular a progenerator for \mathfrak{M}_Λ , there exists $n \in \mathbb{N}$ such that Λ_Λ is a direct summand of $(P^*)^n$. Hence $M = \Lambda \otimes_\Lambda M$ occurs as a direct summand of $F(M)^n = (P^*)^n \otimes_\Lambda M$ and thus is annihilated by x , as desired. \square

Finally, we formulate an analogue of Example 1.2.

Proposition A.13. *Let I be a two-sided ideal of Λ . Under the identification $\Lambda \simeq \text{End}_R(P)$ we have $I = \text{Hom}_R(P, \mathfrak{a} \cdot P)$ for a uniquely determined R -ideal \mathfrak{a} . Then we have an equality*

$$\text{Fitt}_\Lambda(\Lambda/I) = \text{Fitt}_R(\text{Hom}_R(P, R/\mathfrak{a})).$$

Proof. That I is of the given form is [Rei03, Theorem 16.14(v)]. The canonical short exact sequence $0 \rightarrow I \rightarrow \Lambda \rightarrow \Lambda/I \rightarrow 0$ yields a short exact sequence

$$0 \rightarrow P^* \otimes_{\Lambda} I \rightarrow P^* \rightarrow P^* \otimes_{\Lambda} \Lambda/I \rightarrow 0.$$

We claim that the image of $P^* \otimes_{\Lambda} I$ in P^* equals $\mathrm{Hom}_R(P, \mathfrak{a})$. In fact, the map

$$P^* \otimes_{\Lambda} I = \mathrm{Hom}_R(P, R) \otimes_{\Lambda} \mathrm{Hom}_R(P, \mathfrak{a} \cdot P) \longrightarrow P^* = \mathrm{Hom}_R(P, R)$$

is given by $f \otimes g \mapsto f \circ g$, where $f \in \mathrm{Hom}_R(P, R)$ and $g \in \mathrm{Hom}_R(P, \mathfrak{a} \cdot P)$. As the image of g lies in $\mathfrak{a} \cdot P$ and f is R -linear, the image of $f \circ g$ actually lies in $\mathrm{Hom}_R(P, \mathfrak{a})$. Thus we in fact have a map

$$\alpha : P^* \otimes_{\Lambda} I = \mathrm{Hom}_R(P, R) \otimes_{\Lambda} \mathrm{Hom}_R(P, \mathfrak{a} \cdot P) \longrightarrow \mathrm{Hom}_R(P, \mathfrak{a}).$$

To show that α is surjective (and thus an isomorphism), it suffices to show the corresponding statement after localisation at each prime ideal \mathfrak{p} of R . However, a projective module over a local ring is free, so there exists $n \in \mathbb{N}$ such that $P \simeq R^n$. Via this isomorphism, both the domain and codomain of α identify naturally with $\bigoplus_{i=1}^n \mathfrak{a}$ and α becomes the identity map. Hence α is an isomorphism.

We have shown that we have an isomorphism of R -modules

$$P^* \otimes_{\Lambda} \Lambda/I \simeq \mathrm{Hom}_R(P, R)/\mathrm{Hom}_R(P, \mathfrak{a}) \simeq \mathrm{Hom}_R(P, R/\mathfrak{a}),$$

where the last isomorphism holds by projectivity of P . The result now follows. \square

Example A.14. Suppose that P is free of rank n over R for some $n \in \mathbb{N}$. Then Λ naturally identifies with the matrix ring $M_{n \times n}(R)$. Moreover, $I = M_{n \times n}(\mathfrak{a})$ and so $\Lambda/I = M_{n \times n}(R/\mathfrak{a})$. Thus we have $\mathrm{Fitt}_{\Lambda}(\Lambda/I) = \mathfrak{a}^n$ and so Proposition A.13 recovers [JN13, Theorem 2.2(ix)]. One can also view Λ/I as a module over the centre R , but $\mathrm{Fitt}_R(\Lambda/I) = \mathfrak{a}^{n^2}$ is properly contained in $\mathrm{Fitt}_{\Lambda}(\Lambda/I)$ if $n > 1$ (also see Remark A.5).

Acknowledgements. The first named author was supported by EPSRC First Grant EP/N005716/1 ‘Equivariant Conjectures in Arithmetic’. The second named author acknowledges financial support provided by the Deutsche Forschungsgemeinschaft (DFG) within the Heisenberg programme (No. NI 1230/3-1).

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