Markov Automata: Deciding Weak Bisimulation by means of "non-naïvely" Vanishing States

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Abstract. This paper develops a decision algorithm for weak bisimulation on Markov Automata (MA). For that purpose, different notions of vanishing state (a concept known from the area of Generalised Stochastic Petri Nets) are defined. Vanishing states are shown to be essential for relating the concepts of (state-based) naïve weak bisimulation and (distribution-based) weak bisimulation. The bisimulation algorithm presented here follows the partition-refinement scheme and has exponential time complexity.

1 Introduction

Markov Automata (MA) are a powerful formalism for modelling systems with nondeterminism, probability and continuous time. Unfortunately up to now no algorithm for weak bisimulation on MA has been published. The contribution of this paper is a transfer of the notion of vanishing states, as known in the area of Generalised Stochastic Petri Net, to the MA setting. This allows us to state a decision algorithm for weak MA bisimulation.

Since MA have been defined in 2010 [3] it remained an open problem how to decide weak MA bisimilarity. As naïve weak bisimulation on MA corresponds to weak probabilistic bisimulation on Probabilistic Automata (PA), naïve weak MA bisimulation is known to be decidable since 2002 [1]. There, an exponential time algorithm was presented. In 2012 a polynomial time algorithm has been presented for deciding naïve weak MA bisimulation [5]. Our algorithm has exponential time complexity (the result of [5] does not seem to be applicable for the weak case).

The paper is organised as follows: In Sec. 2 we present the necessary preliminaries and recall a mapping from MA to PA from [3]. Sec. 3 recalls some facts on weak bisimulation. In Sec. 4 we define different notions of vanishing states and use them to relate weak bisimulation and naïve weak bisimulation. Finally, Sec. 5 describes our partition refinement algorithm that heavily relies on Sec. 4.

2 Preliminaries

First we define the notion of discrete subdistribution and related terms and notations:

Definition 1 ((Sub-)distributions). *A mapping* $\mu : S \to [0,1]$ *is called (discrete) subdistribution, if* $\sum_{s\in S}\mu(s) \leq 1$. *As usual we write* $\mu(S')$ *for* $\sum_{s\in S'}\mu(s)$. *The* support *of* μ *is defined as* $Supp(\mu) := \{s \in S | \mu(s) > 0\}$. *The empty subdistribution* $\mu_\emptyset$ *is defined by* $Supp(\mu_\emptyset) = \emptyset$. *The* size *of* μ *is defined as* $|\mu| := \mu(S)$. *A subdistribution* μ *is called* distribution *if* $|\mu| = 1$. *The sets* $Dist(S)$ *and* $Subdist(S)$ *denote distributions and subdistributions defined over the set* S. *Let* $\Delta_s \in Dist(S)$ *denote the* Dirac *distribution on* s, *i.e.* $\Delta_s(s) = 1$. *For two subdistributions* μ, μ' *the sum* $\mu'' := \mu \oplus \mu'$ *is defined as* $\mu''(s) := \mu(s) + \mu'(s)$ *(as long as* $|\mu''| \leq 1$. *As long as* $c \cdot |\mu| \leq 1$, *we denote by* $c\mu$ *the subdistribution defined by* $(c\mu)(s) := c \cdot \mu(s)$. *For a subdistribution* μ *and a state* $s \in Supp(\mu)$ *we define* $\mu - s$ *by*

$$(\mu - s)(t) = \begin{cases} \mu(t) & \text{for } t \neq s \\ 0 & \text{for } t = s \end{cases}$$

The definition of Markov Automata we use is the one from [3,4].

Definition 2 (Markov Automata [3]). *A Markov automaton MA is a tuple* $(S, Act, \longrightarrow, \twoheadrightarrow, s_0)$, *where*

- S *is a nonempty finite set of states*
- Act *is a set of actions containing the internal action* τ
- $\longrightarrow \subset S \times Act \times Dist(S)$ *a set of action-labelled probabilistic transitions (PT)*
- $\twoheadrightarrow \subset S \times \mathbb{R}_{\geq 0} \times S$ *a set of Markovian timed transitions (MT) and*
- $s_0 \in S$ *the initial state*

A state in an MA is called stable *if it has no emanating* τ *transitions, otherwise it is called unstable. A stable state* s *will be denoted by* $s{\downarrow}$.

Remark 1. In order to make our decision algorithm feasible we assume in the following that, in contrast to the original definition from [3,4], *all* sets in Definition 2 are finite.

For simplicity we define probabilistic automata (PA) in terms of MA.

Definition 3. *A probabilistic automaton (PA) is a MA* $M = (S, Act, \to, \emptyset, s_0)$.

Note that the PA we define are also PA in the sense of [7], but not vice versa ([7] also allows for different actions within one distribution). We do not need the more general definition of Segala in this paper.

For the mapping from MA to PA introduced in [3] we need to define the probability distribution on successor states. Remark 16 in the appendix explains why this is needed.

Definition 4 (modified version of Definition 3 in [3]). *Let* $M = (S, Act, \longrightarrow, \twoheadrightarrow, s_0)$ *be a MA. Define*

$$rate(s, s') := \sum_{(s,\lambda,s') \in \twoheadrightarrow} \lambda$$

and $rate(s) := \sum_{s' \in S} rate(s, s')$ *which is called the* exit rate *of state* s*. The corresponding probability distributions are defined in the following way:*

$$P_s := \begin{cases} s' \mapsto \frac{rate(s,s')}{rate(s)} & \text{for } rate(s) \neq 0 \\ \Delta_s & \text{otherwise} \end{cases}$$

We would like to stress that this definition in the original setting from [3,4] is problematic, as for infinite sets $\twoheadrightarrow$ the exit rate may not converge.

2.1 A mapping from MA to PA

The remarkable idea of [3] is to define bisimulations on MA using a mapping from MA to PA. The basic ingredient is a set of special actions, denoted by $\chi(.)$, that cover timed behaviour. In the setting of [3,4] countable action sets are mapped to uncountable action sets by definition, as for every real number a new action name is introduced. In order to keep the action set finite, we redefine Act^χ in the context of a fixed MA:

Definition 5. *Let* $M = (S, Act, \longrightarrow, \twoheadrightarrow, s_0)$ *be a MA. Assume* $\forall r \in \mathbb{R}_{\geq 0}\ \chi(r) \notin Act$ *and define* $\mathcal{RT} := \{rate(s) | s \in S\}$ *(which is finite). Then we define* $Act^{\chi(\mathcal{RT})} = \{\chi(r) | r \in \mathcal{RT}\}$ *and* $Act^\chi := Act \cup Act^{\chi(\mathcal{RT})}$.

There is a mapping from MA to PA (adapted from [3]) where we use Definition 4:

Definition 6. *Let* $M = (S, Act, \longrightarrow, \twoheadrightarrow, s_0)$ *be a MA. Define the transitions* $\rightarrow$ *as follows: For* $s \in S$ *define*

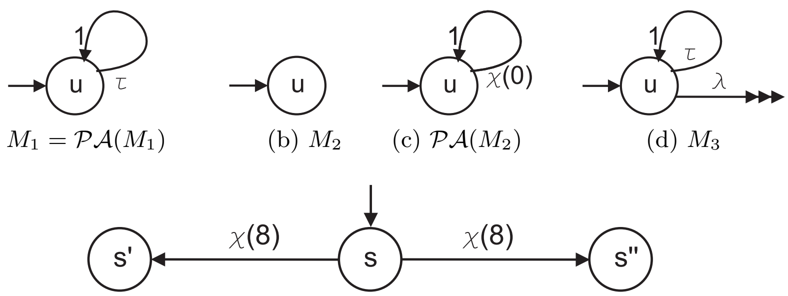
$$s \xrightarrow{\alpha} \mu \text{ if } \begin{cases} \alpha \in Act \text{ and } s \xrightarrow{\alpha} \mu \\ s\downarrow, \alpha = \chi(rate(s)) \in Act^{\chi(\mathcal{RT})} \text{ and } \mu = P_s \end{cases}$$

Then the mapping $\mathcal{PA} : MA \rightarrow PA$ *is defined by* $M \mapsto (S, Act^\chi, \rightarrow, \emptyset, s_0)$.

Note that every timed transition is part of a special $\chi(\cdot)$ action. So the set of actions is increased by the mapping $PA(\cdot)$, but no timed transition remains in the image. For more details on the procedure we refer to [3]. Note further that this is not always well defined. If you have infinitely many Markovian transitions emanating from one state, the exit rate does not need to converge. As a result you cannot define the corresponding probability distribution correctly.

Example 1. Two basic examples are given in Fig. 1. M_1 (Fig. 1a) has a τ loop, so no timed transition (i.e. $\chi(\cdot)$) exists after the transformation, i.e. $\mathcal{PA}(M_1) = M_1$. In the example M_2 (Fig. 1b) u is a stable state and therefore the transformation leads to a χ transition with exit rate 0 (Fig. 1c).

Remark 2. The mapping $\mathcal{PA}(\cdot)$ is not surjective, as a Markovian race condition is always converted to a deterministic χ transition. That means for example the PA in Fig. 1e is not in $\mathcal{PA}(MA)$. The mapping is also not injective, as $M_3 \neq M_1$ in Fig. 1, but $\mathcal{PA}(M_1) = \mathcal{PA}(M_3) = M_1$.



(a) $M_1 = \mathcal{PA}(M_1)$ (b) M_2 (c) $\mathcal{PA}(M_2)$ (d) M_3

(e) PA which is not in $\mathcal{PA}(MA)$

Fig. 1: MA to PA transformations

2.2 Weak transitions

In this section we recall definitions from [3,4], complete them and correct minor typos.

Definition 7 (Sequences of integers). *Let* $(\mathbb{Z}^+)^*$ *the set of all possibly infinite sequences of positive integers. For* σ, $\sigma' \in (\mathbb{Z}^+)^*$ *we write* $\sigma \leq \sigma'$ *if there exists a (possibly empty)* $\phi \in (\mathbb{Z}^+)^*$ *such that, by usual concatenation,* $\sigma\phi = \sigma'$*. If the equality holds for nonempty* ϕ*, we also write* $\sigma < \sigma'$

Definition 8 (Labelled trees, cf. Section II in [3], Section 2 in [4]). *Let* $\mathcal{A} \subseteq (\mathbb{Z}^+)^*$ *and* L *be an arbitrary set. A function*

$$\mathcal{T} : \mathcal{A} \to L$$

is called (infinite) L-labelled tree*, if the following conditions hold:*

1. $(\sigma' \in \mathcal{A}$ *and* $\sigma \in (\mathbb{Z}^+)^*$ *and* $\sigma \leq \sigma')$ $\Rightarrow \sigma \in \mathcal{A}$
2. $(\sigma i \in \mathcal{A}$ *and* $i > 1) \Rightarrow \sigma(i-1) \in \mathcal{A}$
3. $\varepsilon \in \mathcal{A}$ *(empty sequence)*

The elements of $\sigma \in \mathcal{A}$ *are called* nodes *of* $\mathcal{T}$*. An element* $\sigma \in \mathcal{A}$ *is called* leaf *of* $\mathcal{T}$ *if there is no* $\sigma' \in \mathcal{A}$ *such that* $\sigma < \sigma'$*. The symbol* ε *is called the* root *of* $\mathcal{T}$*. The set of leaves of* $\mathcal{T}$ *is denoted by* $Leaf_{\mathcal{T}}$ *and the set of inner nodes by* $Inner_{\mathcal{T}}$*. If* $\mathcal{A} = \{\varepsilon\}$ *we define* $Leaf_{\mathcal{T}} = Inner_{\mathcal{T}} = \varepsilon$*, otherwise we define* $Inner_{\mathcal{T}} := \mathcal{A} \setminus Leaf_{\mathcal{T}}$*. For a node* σ *we define* $Children(\sigma) = \{\sigma i | \sigma i \in \mathcal{A}\}$*.*

In the following we only consider L-labelled trees with *finite branching*, i.e. for all $\sigma \in \mathcal{A} : |Children(\sigma)| < \infty$. Next we will consider $(S \times \mathbb{R}_{\geq 0} \times (Act^{\chi} \dot{\cup} \{\perp\}))$-labelled trees $\mathcal{T} : \mathcal{A} \to (S \times \mathbb{R}_{\geq}0 \times (Act^{\chi} \cup \{\perp\}))$. We introduce abbreviations for the projections on the different components of the label (π_i being the projection on the i-th component): $Sta_{\mathcal{T}} := \pi_1 \circ \mathcal{T}$, $Prob_{\mathcal{T}} := \pi_2 \circ \mathcal{T}$ and $Act_{\mathcal{T}} := \pi_3 \circ \mathcal{T}$.

Definition 9 (Transition Tree, cf. Definition 6 in [3,4]). *Let M be a MA and $\mathcal{PA}(M) = (S, Act^\chi, \rightarrow, \emptyset, s_0)$. A* transition tree *$\mathcal{T}$ over M is a $(S \times \mathbb{R}_{\geq} 0 \times (Act^\chi \dot\cup \{\perp\}))$ labelled tree that satisfies the following conditions:*

1. $Prob_\mathcal{T}(\varepsilon) = 1$,
2. $\forall \sigma \in Leaf_\mathcal{T} : Act_\mathcal{T}(\sigma) = \perp$
3. $\forall \sigma \in Inner_\mathcal{T} \setminus Leaf_\mathcal{T} : \exists \mu : Sta_\mathcal{T}(\sigma) \stackrel{Act_\mathcal{T}(\sigma)}{\rightarrow} \mu$ *($\rightarrow$ being the transitions of $\mathcal{PA}(M)$) and* $Prob_\mathcal{T}(\sigma) \cdot \mu = \oplus_{\sigma' \in Children_\mathcal{T}(\sigma)} Prob_\mathcal{T}(\sigma') \cdot \Delta_{Sta_\mathcal{T}(\sigma')}$

An internal *transition tree over M is a transition tree where $\forall \sigma \in Inner_\mathcal{T} \setminus Leaf_\mathcal{T} : Act_\mathcal{T}(\sigma) = \tau$. Whenever the context is clear, we omit the term "over M". The distribution associated to a transition tree $\mathcal{T}$ is defined as $\mu_\mathcal{T} := \oplus_{\sigma' \in Leaf_\mathcal{T}} Prob_\mathcal{T}(\sigma') \cdot \Delta_{Sta_\mathcal{T}(\sigma')}$. We say that $\mu_\mathcal{T}$ is* induced *by $\mathcal{T}$.*

Now we can define weak transitions:

Definition 10 (Weak Transition, cf. Definition 7 in [3,4]). *Let M be an MA, $\mathcal{PA}(M) = (S, Act^\chi, \rightarrow, \emptyset, s_0)$ and $s \in S$. We define weak transitions by looking at transition trees over M. We write:*

- *$s \Rightarrow \mu$ if μ is induced by some internal transition tree $\mathcal{T}$ over M with $Sta_\mathcal{T}(\varepsilon) = s$.*
- *$s \stackrel{\alpha}{\Rightarrow} \mu$ if μ is induced by some transition tree $\mathcal{T}$ over M with $Sta_\mathcal{T}(\varepsilon) = s$, where on every path from the root to a leaf at least for one node σ it holds that $Act_\mathcal{T}(\sigma) = \alpha$. In case that $\alpha \neq \tau$ there must be exactly one such node on each of these paths. All other inner nodes must be labelled by τ.*
- *$s \stackrel{\hat\alpha}{\Rightarrow} \mu$ if ($\alpha = \tau$ and $s \Rightarrow \mu$) or ($\alpha \neq \tau$ and $s \stackrel{\alpha}{\Rightarrow} \mu$)*

The difference between the $\hat\alpha$ and α notation is only in the case $\alpha = \tau$: A transition labelled with $\hat\tau$ can also remain in the same state without performing any transition, while label τ requires at least one τ transition to be performed. Next we define the notation of combined transitions:

Definition 11 (Combined Transitions, cf. Definition 8 in [3,4]). *Let M be an MA, $\mathcal{PA}(M) = (S, Act^\chi, \rightarrow, \emptyset, s_0)$ and $s \in S$. We write $s \stackrel{\alpha}{\Rightarrow}_C \mu$, if $\alpha \in Act^\chi$ and there is a finite set I such that such that $s \stackrel{\alpha}{\Rightarrow} \mu_i$ for all $i \in I$ and μ is a* convex combination *of $\{\mu_i\}_{i \in I}$, i.e. $\exists \{c_i \in \mathbb{R}^+\}_{i \in I} : \sum_{i \in I} c_i = 1, \mu = \oplus_{i \in I} c_i \mu_i$. Combined $s \stackrel{\hat\alpha}{\Rightarrow}_C \mu$ transitions are defined similarly.*

The above notations may be generalised from states to subdistributions:

Definition 12 (Transitions between Subdistributions, cf. Definition 9 in [3,4]). *Let S be a set of staes, $\mu \in Subdist(S)$, $\rightsquigarrow \in \{\rightarrow, \Rightarrow, \hat\Rightarrow, \Rightarrow_C, \hat\Rightarrow_C\}$. We write $\mu \rightsquigarrow \gamma$ if for all $s_i \in Supp(\mu)$ it holds that $s_i \rightsquigarrow \gamma_i$ and $\gamma = \oplus_{s_i \in Supp(\mu)} \mu(s_i) \gamma_i$.*

3 Relating naïve weak & weak bisimulation

Remember that for an MA M its transitions have been defined by means of $\mathcal{PA}(M)$, so in the following it is safe to assume that all Markov Automata are represented by their PA images. All calculations will be made in this context.

The first definition is a necessary adaptation of the definition in [3,4] that better corresponds to the definition of [8]. We use $\stackrel{\hat{\alpha}}{\Rightarrow}$ instead of $\stackrel{\alpha}{\Rightarrow}$ because then it is allowed for the "defender" to remain in its state even if there is no explicit τ loop (this corrected definition has also been given in [2]).

Definition 13 (Naïve weak bisimulation in the spirit of [8]). *An equivalence relation $\mathcal{R}$ on the set of states S of an MA $M = (S, Act, \longrightarrow\!\!\!\!\!\bullet, \twoheadrightarrow, s_0)$ is called* naïve weak bisimulation *if and only if $x\mathcal{R}y$ implies for all $\alpha \in Act^{\chi}$: $(x \stackrel{\alpha}{\rightarrow} \mu)$ implies $(y \stackrel{\hat{\alpha}}{\Rightarrow}_C \mu')$ with $\mu(C) = \mu'(C)$ for all $C \in {}^{S}\!/\!_{\mathcal{R}}$ (note that the transitions are regarded in $\mathcal{PA}(M)$). Two MA are called* naïvely weakly bisimilar *if their initial states are related by a naïve weak bisimulation relation on the direct sum of their states.*

We would like to note that modulo naïve weak bisimulation it is possible to omit τ-loops in the image $\mathcal{PA}(\cdot)$. The property whether a state is stable or unstable can still be recovered by looking for the presence (or absence) of χ transitions.

As naïve weak bisimulation for MA is defined by means of the transformation $\mathcal{PA}(\cdot)$, by comparing definitions the following lemma is clear.

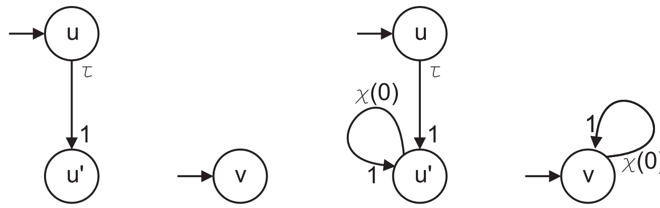
Lemma 1. *Let MA be the set of Markov Automata. Naïve weak MA bisimulation corresponds to weak probabilistic bisimulation on $\mathcal{PA}(MA)$ (as defined in [8]).*

Remark 3. This lemma does not hold for the original definition of naïve weak bisimulation in [3,4]. A minimal example is given in Fig. 2. According to the definition of [3,4], M_1 and M_2 are not bisimilar (as M_2 cannot perform a τ transition). In weak probabilistic bisimulation we have that $\mathcal{PA}(M_1)$ and $\mathcal{PA}(M_2)$ are bisimilar, as a τ loop is implicitly assumed at every state.

[3,4] argued that the (state-based) notion of naïve weak bisimulation is too fine. Therefore they defined the coarser notion of (distribution-based) weak bisimulation:

Definition 14 (Weak bisimulation [3]). *A relation $\mathcal{R}$ on sub-distributions over a set of states S of an MA $M = (S, Act, \longrightarrow\!\!\!\!\!\bullet, \twoheadrightarrow, s_0)$ is called* weak bisimulation *if for all $(\mu_1, \mu_2) \in \mathcal{R}$ it holds that (transitions regarded in $\mathcal{PA}(M)$)*

A.) $|\mu_1| = |\mu_2|$
B.) $\forall s \in Supp(\mu_1), \forall \alpha \in Act^{\chi} : \exists \mu_2^g, \mu_2^b : \mu_2 \Rightarrow_C \mu_2^g \oplus \mu_2^b$ *such that*
  (i) $(\mu_1(s) \cdot \Delta_s)\mathcal{R}\mu_2^g$ *and* $(\mu_1 - s)\mathcal{R}\mu_2^b$
  (ii) $(s \stackrel{\alpha}{\rightarrow} \mu_1') \Rightarrow (\exists \mu'' : \mu_2^g \stackrel{\hat{\alpha}}{\Rightarrow}_C \mu''$ *and* $(\mu_1(s) \cdot \mu_1')\mathcal{R}\mu'')$



(a) M_1 (b) M_2 (c) $\mathcal{PA}(M_1)$ (d) $\mathcal{PA}(M_2)$

Fig. 2: MA to PA transformations

C.) a symmetric condition with μ_1 and μ_2 interchanged (roles of left-hand side and right-hand side also interchanged)

Two distributions μ, γ are called weakly bisimilar *(with respect to some MA M), written $\mu \approx \gamma$, if the pair (μ, γ) is contained in a weak bisimulation relation (with respect to M). Two states are called weakly bisimilar if their corresponding Dirac distributions are weakly bisimilar. We write $s \approx_\Delta t$ for $\Delta_s \approx \Delta_t$. Two MA are called weakly bisimilar if their initial states are weakly bisimilar in the direct sum of the MA.*

Remark 4. Theorem 1 in [3,4] shows that $\approx$ is an equivalence relation (and therefore also $\approx_\Delta$). So it is possible to talk about quotients. An equivalence class of a state s in $S/\approx_\Delta$ is denoted by $[s]_{\approx_\Delta}$. When the context is clear we will only write $[s]$ without mentioning the quotient.

A deep Theorem is Theorem 2 in [3,4]. The problem is that it is trivially wrong in the way it is written down there, as the following Remark shows.

Remark 5. Look at the automaton given in Fig. 3. According to Def. 14 we clearly have $A \approx B$, but Theorem 2 in [3,4] tells us that $A \not\approx B$ as $B \overset{\hat{\tau}}{\not\Rightarrow} \mu$ with $\mu \approx \Delta_B$. Moreover, the one-way condition stated in part 1 of Theorem 2 in [3,4] is not sufficient, because it does not imply that $\approx_\Delta$ is an equivalence relation.

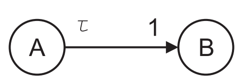


Fig. 3: Counterexample to Theorem 2 in [4]

To face the issues raised in Remark 5 we state Theorem 1.

Theorem 1 (Corrected version of Theorem 2 in [3,4]). *Let $M = (S, Act, \dashrightarrow, \twoheadrightarrow, s_0)$ be a MA and $PA(M) = (S, Act^\chi, \rightarrow, \emptyset, s_0)$ its corresponding PA. Let further be E, $F \in S$ and μ, $\gamma \in Dist(S)$. Then*

1. $E \approx_\Delta F$ *iff the following two implications hold:*
   (a) *whenever* $E \xrightarrow{\alpha} \mu$ *then for some* $\gamma : F \stackrel{\hat{\alpha}}{\Rightarrow}_C \gamma$ *and* $\mu \approx \gamma$
   (b) *whenever* $F \xrightarrow{\alpha} \gamma$ *then for some* $\mu : E \stackrel{\hat{\alpha}}{\Rightarrow}_C \mu$ *and* $\gamma \approx \mu$
2. $\mu \approx \gamma$ *iff* $\exists \mu', \gamma' : \mu \Rightarrow \mu', \gamma \Rightarrow \gamma', \mu \approx \mu', \gamma \approx \gamma'$ *and* $\forall C \in S/_{\approx_\Delta} : \mu'(C) = \gamma'(C)$

A direct consequence of the Theorem is the following corollary (already indicated in [4])

Corollary 1. *If two MA* M_1 *and* M_2 *are naïvely weakly bisimilar, they are weakly bisimilar.*

Proof. Immediate by Def. 13 and Theorem 1: Simply take $\mu' := \mu$ and $\gamma' := \gamma$

Remark 6. Combined transitions $\Rightarrow_C$ can be used to rescale loops. A basic example is given in Fig. 4. An infinite transition tree rooted at s leads to the distribution $\frac{1}{2}\Delta_x \oplus \frac{1}{2}\Delta_y$. To mimic the τ transition of s, t has to perform a transition combined of $\frac{2}{3}$ times $t \rightarrow \frac{1}{2}\Delta_x \oplus \frac{1}{2}\Delta_y$ and $\frac{1}{3}$ times $t \Rightarrow \Delta_t$. Without using combined transitions t could not mimick this transition of s.

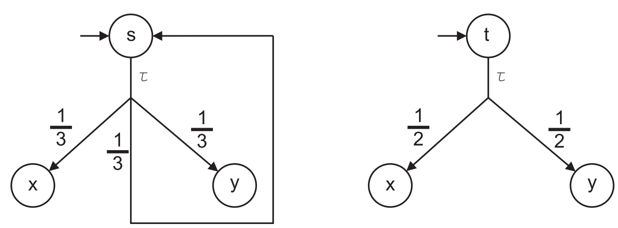


Fig. 4: Resolving τ loop

4 Vanishing states and their elimination

We now introduce a notion of vanishing states in the context of MA.

Definition 15 (local change of transitions). *Let* $P = (S, Act, \rightarrow, \emptyset, s_0)$ *be a PA,* $s \in S$*,* s *unstable. We denote the transitions emanating from* s *by*

$$\mathcal{T}_s := \{(s, \alpha, \mu) | \exists \mu \in Dist(S), \alpha \in Act : s \xrightarrow{\alpha} \mu\}.$$

We define the set of weak combined transitions in P *emanating from* s *as* $\mathfrak{C}(s) := \{(s, \tau, \nu) \in \{s\} \times \{\tau\} \times Dist(S) | s \Rightarrow_C \nu\}$*. For any* $\mathcal{T}_s^{new} \subseteq \mathfrak{C}(s)$ *we define the PA*

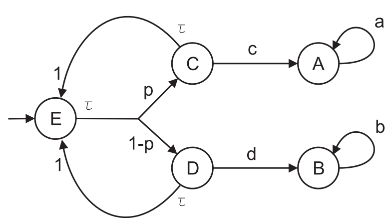
$$P(\mathcal{T}_s^{new}) := (S, Act, (\rightarrow \setminus \mathcal{T}_s) \cup \mathcal{T}_s^{new}, \emptyset, s_0)_{s \mapsto s'}$$

where we rename s *to* s' *(assuming that* s' *is a new symbol). If* $\mathcal{T}_s^{new} = \{(s, \tau, \nu)\}$ *we also write* $P_{(s,\nu)}$ *— or simpliy* P' *if the context is clear — instead of* $P(\{(s, \tau, \nu)\})$*.*

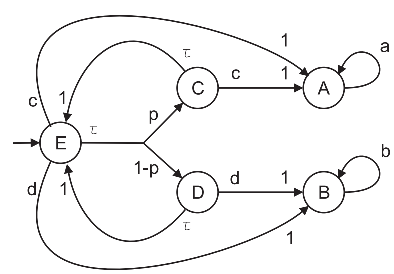
Definition 16 (vanishing state). *Let $P = (S, Act, \rightarrow, \emptyset, s_0)$ be a PA. Let $s \in S$ be unstable and $\mathcal{T}_s$ as in Def. 15. State s is called*

trivially vanishing *if $\mathcal{T}_s = \{(s, \tau, \nu)\}$.*
vanishing *if there exists $(s, \tau, \nu) \in \mathfrak{C}(s)$ such that $s \approx_\Delta s'$ when comparing P and $P_{(s,\nu)}$. In this case $P_{(s,\nu)}$ — or (s, ν), for short — is called a* vanishing representation *of s.*
non-naïvely vanishing *if it is vanishing and there is a vanishing representation $P_{(s,\nu)}$ with $\exists x_0 \in Supp(\nu)$: $s \not\approx_\Delta x_0$. (*nn-vanishing*, for short)*

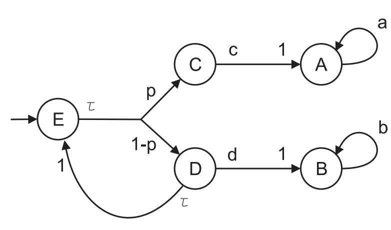
Example 2. Assume that $p \in (0, 1)$. State E in Fig. 5a is trivially vanishing since it only has an emanating τ transition. A vanishing (but not nn-vanishing) state E is given in Fig. 5b. Note that all non-τ transitions emanating from E may be omitted as they can be mimicked by appropriate weak transitions. The automaton in Fig. 5a is the corresponding vanishing representation. E is naïvely vanishing as it turns out to be in the same class as C and D modulo weak bisimulation. For the last example, first note that in Fig. 5c C and D cannot be weakly bisimilar (because C can only perform the c to A, while D can additionally perform the d to B). As E is trivially vanishing we notice that it is also nn-vanishing, because E moves to the distribution $p\Delta_C \oplus (1-p)\Delta_D$, where $C \not\approx D$. Moreover, since in Fig. 5c D is not vanishing (and E is nn-vanishing), we have that $E \not\approx D$.



(a) E trivially vanishing



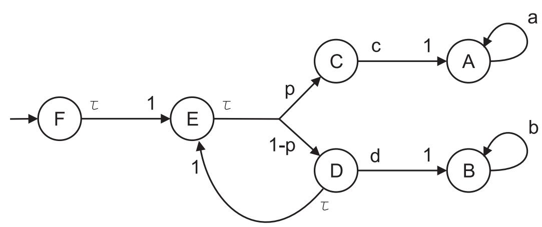
(b) E vanishing



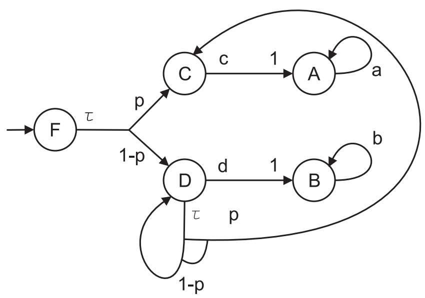
(c) E nn-vanishing

Fig. 5: Examples of vanishing states

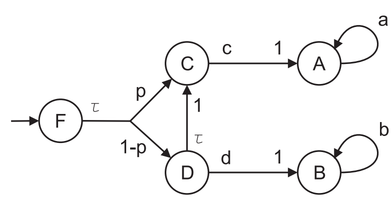
Example 3. Assume again that $p \in (0,1)$. The example in Fig. 6a shows that the property of being nn-vanishing is preserved by weak bisimilarity. This means that nn-vanishing states may be grouped into vanishing classes modulo weak bisimulation. We saw in the example before that E is an nn-vanishing state. Obviously E and F belong to the same class modulo weak bisimulation. We want to know what the vanishing representation for F is. Now we ignore E, cf. Fig. 6b (Definition 17 and Lemma 3 will justify that this is reasonable), and rescale the τ transitions emanating from D according to Remark 6 (cf. Fig. 6c). We see that there is a τ connection from F to states C and D and it holds that $\varDelta_F \approx p \cdot \varDelta_C \oplus (1-p) \cdot \varDelta_D$. So we use $\mathcal{T}_s^{new} = \{(F, \tau, p \cdot \varDelta_C \oplus (1-p) \cdot \varDelta_D)\}$ to get the vanishing representation for F. Note also that in Fig. 6a we need two τ steps to get the weak transition that can be used as vanishing representation of F. This indicates that it is not trivial to find vanishing representations.



(a) E and F nn-vanishing



(b) E ignored



(c) D rescaled

Fig. 6: Examples of nn-vanishing states

Remark 7. As naïve weak bisimulation is defined on states rather than distributions it is easy to see that a nn-vanishing state s cannot be vanishing in the naive sense: only if *all* successor states still lay in the class of s it would be possible, as the distribution then could be identified as one single state.

The use of combined transitions $\mathfrak{C}(s)$ for vanishing representations is too general, as the following lemma shows:

Lemma 2. *Every vanishing representation* (s, μ) *can be transformed to a (non-combined) vanishing representation* (s, γ') *where* $s \Rightarrow \gamma'$ *is induced by a Dirac determinate scheduler in the sense of [1].*

Proof. Look at the diagram in Fig. 7. Firstly we show that we can restrict ourselves to the non-combined case. Let $P_{(s,\mu)}$ be the vanishing representation of s (so, by definition, $\Delta_s \approx \mu$). If the corresponding transition is not combined, we are done. So we may assume that the vanishing representation is given by a combined transition (indicated by the C in the upper arrow). The down arrows are granted by part 2 of Theorem 1 (also $\Delta_s \approx \gamma'$ and $\mu \approx \mu'$). The identity in

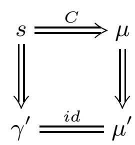


Fig. 7: Finding non-combined vanishing representations

the lower line is on classes $C \in S/{\approx_\Delta}$ (as described in part 2 of Theorem 1). By transitivity of $\approx$ it is clear that μ' is also a vanishing representation. But then we also may take γ' as the vanishing representation, which is reached by a single transition tree. This construction also works with Dirac determinate schedulers (observe that in the proof of Lemma 16 in [4] Dirac determinate schedulers have been used to construct the transition trees), so the lemma is shown.

Corollary 2. *Let* $P = (S, Act, \rightarrow, \emptyset, s_0)$ *be a PA. State* $s \in S$ *is nn-vanishing if and only if there is a vanishing representation of* s *generated by a Dirac determinate scheduler that leaves the equivalence class of* s *in* $S/{\approx_\Delta}$ *with positive probability.*

Proof. Just a reformulation of Definition 16 using Lemma 2.

Definition 17 (Elimination of vanishing states). *Let* $P = (S, Act, \rightarrow, \emptyset, s_0)$ *be a PA. Let* $s \in S$ *be a vanishing state and let* $s \overset{\tau}{\rightarrow}' \nu$ *be the only transition emanating from* s *in the vanishing representation* $P' = P_{(s,\nu)} = (S, Act, \rightarrow', \emptyset, s_0)$*. The elimination of* s *is defined by two steps:*

1. *Rescaling (cf. Remark 6):*

$$\rightarrow'_{res} = \begin{cases} \rightarrow' \setminus \{(s, \tau, \nu)\} & \text{if } \nu = \Delta_s \\ (\rightarrow' \setminus \{(s, \tau, \nu)\}) \,\dot{\cup}\, \{(s, \tau, \frac{1}{1-\nu(s)}(\nu - s))\} & \text{otherwise} \end{cases}$$

2. *Elimination (only performed if after rescaling a transition* $s \xrightarrow{\tau}'_{res} \nu$ *remains):*

$$P'^{\widehat{s}} = \begin{cases} (S \setminus \{s\}, Act, \rightarrow'', \emptyset, s_0) & \text{if } s \neq s_0 \\ ((S \setminus \{s_0\}) \dot\cup \{s_0^\circ\}, Act, \rightarrow'' \dot\cup \{s_0^\circ \xrightarrow{\tau}'_{res} \nu\}, \emptyset, s_0^\circ) & \text{if } s = s_0 \text{ and} \\ & \exists t \rightarrow'_{res} \nu : s_0 \in Supp(\nu) \\ P' & \text{otherwise} \end{cases}$$

where $\rightarrow'' := \{(t, \alpha, \mu') | t \xrightarrow{\alpha}'_{res} \mu, t \in S \setminus \{s\}, \mu' := \mu_{s \rightarrow \nu}\}$*. Here* $\mu_{s \rightarrow \nu}$ *denotes the replacement of every occurrence of* s *by the corresponding distribution* ν*: Without loss of generality let* μ *be of the form* $\mu := c_s \Delta_s \oplus (\oplus_{i \in I, s_i \neq s} c_i \Delta_{s_i})$ *and* ν *be of the form* $\nu = \oplus_{j \in J, s_j \neq s} d_j \Delta_{s_j}$*. Then we define* $\mu_{s \rightarrow \nu} := c_s(\oplus_{j \in J, s_j \neq s} d_j \Delta_{s_j}) \oplus (\oplus_{i \in I, s_i \neq s} c_i \Delta_{s_i})$*.*

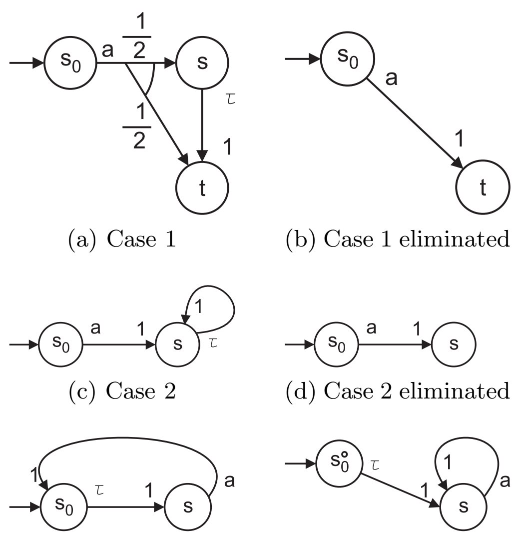
Remark 8. We omit the (s, τ, Δ_s) transition from the set of transitions for the purpose of minimality of the resulting PA. One could also just add the loop case to the case where P' is not changed. Note that even when loops are removed, all information about the MA may be safely recovered. Such a state without loop is a deadlock in PA and no longer vanishing according to our definition (note that there cannot be any other competing transition, as we then could not get the vanishing representation with the τ loop). Looking back to the MA setting it is clear that s must be an unstable state as it does not have the $\chi(0)$ transition.

Example 4. To explain Definition 17 we give the following examples. The first case in the definition is the most common one (cf. Fig. 8a): The vanishing state s is neither the initial state nor does it have a probability-one-self-loop. Therefore the elimination is straight-forward: Redirect all incoming arcs according to the vanishing representation (cf. Fig. 8b). The next case is the probability-one-self-loop case (cf. Fig. 8a): It does not matter whether the vanishing state s is the initial state or not, the self-loop is removed (cf. 8d). In the third case we have a vanishing initial state with incoming transition(s) (cf. Fig. 8e). We add a copy s_0° of the initial state and eliminate the *old* initial state s_0 (cf. Fig. 8f). The last case in Definition 17 is the only remaining one: There is a vanishing initial state but it has no incoming transitions. Then nothing is changed.

Lemma 3 (Elimination does not destroy weak bisimilarity). *For every vanishing state* s *it holds that* $P \approx P'^{\widehat{s}}$

Proof. By definition $P \approx P'$. With the same arguments as in [3] (proof of Theorem 7) it follows that $P' \approx P'^{\widehat{s}}$. So by transitivity of $\approx$ the claim follows.

Remark 9. For a given PA P the set of vanishing states is a superset of the set of trivially vanishing states. By definition, every non-trivially vanishing state s must have a vanishing representation where s is trivially vanishing. So, by successively replacing the P by vanishing representations, only trivially vanishing states remain.



(a) Case 1 (b) Case 1 eliminated

(c) Case 2 (d) Case 2 eliminated

(e) Case 3 (f) Case 3 eliminated

Fig. 8: Different cases of eliminations

The machinery up to now motivates the following definition:

Definition 18. *Let* $P = (S, Act, \to, \emptyset, s_0)$. *Let* $S^v = \{s_1^v, \ldots, s_n^v\}$ *be the set of vanishing states. Denote by* $\widehat{P}$ *the* complete elimination *of* P, *i.e.* $\widehat{P} := (\ldots(P'^{\widehat{s_1^v}})'^{\widehat{s_2^v}}\ldots)'^{\widehat{s_n^v}}$. *Let* $\widehat{P}^*$ *denote the elimination of all nn-vanishing states.*

From this we get the following relation:

Lemma 4 (Complete eliminations and bisimilarity). *For two PA* P_1 *and* P_2 *it holds:* $P_1 \approx P_2 \Leftrightarrow \widehat{P}_1 \approx \widehat{P}_2 \Leftrightarrow \widehat{P}_1^* \approx \widehat{P}_2^*$

Proof. By Lemma 3 we know that elimination preserves weak bisimilarity. The following diagrams show this by the right arrows.

$$\begin{array}{ccccc} P_1 & \xrightarrow{\approx} & \widehat{P}_1^* & \xrightarrow{\approx} & \widehat{P}_1 \\ \downarrow & & \downarrow & & \downarrow \\ P_2 & \xrightarrow{\approx} & \widehat{P}_2^* & \xrightarrow{\approx} & \widehat{P}_2 \end{array}$$

As soon as one of the down arrows is a weak bisimulation, by transitivity of weak bisimulation we immediately get that the other two arrows are also weak bisimulations.

Remark 10. In every example from [3], elimination leads to isomorphic automata (assuming that we replace vanishing initial states by their vanishing representation).

It is clear by Corollary 1 that $\widehat{P}_1 \approx_{\text{naïve}} \widehat{P}_2 \Rightarrow \widehat{P}_1 \approx \widehat{P}_2$ and therefore, by Lemma 4 $P_1 \approx P_2$. Now we try to understand why it is also the case that $P_1 \approx P_2 \Rightarrow \widehat{P}_1 \approx_{\text{naïve}} \widehat{P}_2$.

Lemma 5. *It is in general not possible to eliminate all unstable states.*

Proof. The simplest example possible is the one given in Fig. 9. Assume there was a weak bisimulation relation $\mathcal{R} = \{(\Delta_s, \Delta_t), \ldots\}$, where $(\Delta_E, \Delta_F) \notin \mathcal{R}$ and $(\Delta_F, \Delta_E) \notin \mathcal{R}$. From the definition of weak bisimilarity we would deduce that also $(\Delta_{s_1}, c\Delta_E \oplus (1-c)\Delta_F) \in \mathcal{R}$ for some $c \in [0,1]$. But now observe that we would need to have both $(\Delta_E, c\Delta_E \oplus (1-c)\Delta_F)$ and $(\Delta_F, c\Delta_E \oplus (1-c)\Delta_F)$, therefore $c = 0$ and $c = 1$, which is a contradiction.

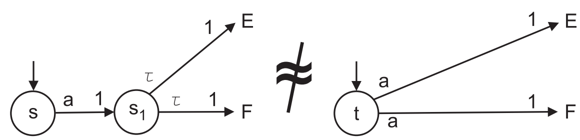


Fig. 9: Lifting nondeterminism (wrong)

Lemma 6. *Only vanishing states can be eliminated*

Proof. Let P be the MA of interest. Suppose that a state s is not vanishing, then for all $(s, \tau, \nu) \in \mathfrak{C}(s)$ we have $P \not\approx P(s, \tau, \nu)$. Using the notation of Def. 16 and taking $\mathcal{T}_s^{new} \subseteq \mathfrak{C}(s)$ (with $|\mathcal{T}_s^{new}| \geq 2$) and assume that $P(\mathcal{T}_s^{new}) \approx P$, but then – as there does not exist a vanishing representation – in s at least two nondeterministic choices must be possible that lead to different behaviour. So with the same argument as in the proof of Lemma 5 we conclude that s cannot be eliminated.

Theorem 2. *It holds that* $P_1 \approx P_2 \Leftrightarrow \widehat{P}_1^* \approx_{\textit{naïve}} \widehat{P}_2^*$.

Proof. $\Leftarrow$ is immediate by Corollary 1 and Lemma 4.
$\Rightarrow$ From Lemma 4 we already know $P_1 \approx P_2 \Leftrightarrow \widehat{P}_1^* \approx \widehat{P}_2^*$. So it remains to show that $\widehat{P}_1^* \approx \widehat{P}_2^*$ is already a naïve weak bisimulation. By part 1 of Theorem 1 we must have for a pair of states $s \approx_\Delta t$ that for every $s \xrightarrow{a} \mu$ we find $t \xRightarrow{\hat{a}} \gamma$ with $\mu \approx \gamma$ (and vice versa). Assume that we split μ according to its support: $\mu = \oplus_{i \in I} c_i \Delta_{s_i}$, then with Lemma 11 of [4] we get a transition $\gamma \Rightarrow_C \gamma' = \oplus_{i \in I} c_i \gamma_i'$ with $\Delta_{s_i} \approx \gamma_i'$. By part 2 of Theorem 1 we get the following diagram for every $i \in I$:

$$\begin{array}{ccc} \Delta_{s_i} & \overset{\approx}{\cdots\cdots} & \gamma_i' \\ \Big\Downarrow \approx & & \Big\Downarrow \approx \\ \mu_i' & \overset{id}{=\!=\!=} & \gamma_i'' \end{array}$$

The identity in the lower line is on classes $C \in S/\approx_\Delta$. By assumption s_i cannot be nn-vanishing (only the initial state could be, but $s_i \neq s_0$ by definition of the elimination), so there cannot be a vanishing representation leaving the equivalence class of s_i, in other words for every $i \in I$: $\mu'_i(C) = \Delta_{s_i}(C) = \gamma''_i(C)$ for all $C \in S/\approx_\Delta$. Defining $\gamma'' = \oplus_{i \in I} c_i \gamma''_i$ we get $t \stackrel{\hat{a}}{\Rightarrow} \gamma \Rightarrow_C \gamma' \Rightarrow \gamma''$, that is $t \stackrel{\hat{a}}{\Rightarrow}_C \gamma''$ and $\forall C \in S/\approx_\Delta : \mu(C) = \gamma''(C)$.

We can use the same argumentation for $t \stackrel{a}{\to} \gamma$ to find $s \stackrel{\hat{a}}{\Rightarrow}_C \mu''$ with $\forall C \in S/\approx_\Delta : \gamma(C) = \mu''(C)$ and we conclude that $\approx_\Delta$ is already a naïve weak bisimulation relation.

Corollary 3. *It holds that* $P_1 \approx P_2 \Leftrightarrow \widehat{P}_1 \approx_{naïve} \widehat{P}_2$.

Proof. $\Leftarrow$ is immediate by Corollary 1 and Lemma 4.
$\Rightarrow$ The hard part is Theorem 2, as non-naïvely vanishing states may be ignored and therefore some equivalence classes modulo weak bisimulation may be ignored. As naïvely vanishing states by definition cannot change their equivalence class modulo weak bisimulation, eliminiation will not change anything – the equivalence class remains and cannot be ignored.

5 A partition refinement algorithm

With Corollary 2 we can find tangible states and Theorem 2 reduces the problem to naïve weak bisimulation. Even if the problem of deciding naïve weak bisimulation as recently been shown to be solvable in polynomial time [5], the decision problem for weak bisimulation remains exponential as finding vanishing representations involves *all* dirac determinate schedulers (which are exponentially many, as has been shown in [1]). For the description of the partition refinement algorithm below we need the convex sets $S(x, \alpha) \subseteq \mathbb{R}^n$ introduced by [1]:

Example 5. For the MA given in Fig. 9 (left-hand side) the resulting set $S(s_1, \tau)$ is given by the shaded triangle in Fig. 10. This triangle encodes all distributions that are reachable via a (weak) combined τ transition starting from s_1, for details on how to calculate those sets we refer to [1].

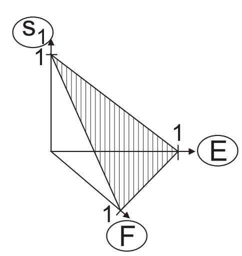


Fig. 10: $S(s_1, \tau)$

Remark 11. In [1] it is shown that the convex sets are the convex hull (CHull) of distributions (i.e. points in $\mathbb{R}^n$) generated by Dirac determinate schedulers. It is shown there that the extremal points (i.e. generators) of the convex hull can be found by a linear program and that the complexity of calculating the sets $S(s, \alpha)$ is exponential for the weak case. Therefore our algorithm also has exponential complexity.

Remark 12 (Restriction of convex sets). The sets $S(s, \alpha) \subseteq \mathbb{R}^n$ may be restricted. We define

$$S(s, \alpha)|_{x_i=0} := \{(x_1, \ldots, x_n) \in S(s, \alpha) | x_i = 0\}$$

Of course, multiple restrictions can also be realised. Especially, we define a restriction to a set of "tangible" states $S(s, \alpha)|_{\mathcal{S}_{tangible}}$ by requiring that $x_i = 0$ for all nn-vanishing states x_i. Note that the restriction of convex sets is still convex by definition.

Our algorithm works according to the partition refinement principle – looking at restrictions of convex sets – as follows:

1. Start with the initial partition $\mathcal{W} = \mathcal{S}_1 \cup \mathcal{S}_2$.
2. For all states s and actions a calculate the convex sets $S(s, \alpha)$ (cf. [1]) and and for every (Dirac determinate) weak transition (s, τ, ν) calculate $S_\nu(s, \alpha)$ ($S_\nu(s, \alpha)$ denotes the convex set calculated for the PA $(P_1 \cup P_2)_{(s,\nu)}$, that is the direct sum of automata P_1 and P_2 where we move to the vanishing representation (s, ν)).
3. Set $\mathcal{S}_{tangible} = \emptyset$.
4. For all states s that are not in $\mathcal{S}_{tangible}$
   - Check whether s can leave its equivalence class in $\mathcal{W}$ with some probability > 0 by a (Dirac determinate) weak transition (s, τ, ν) such that (modulo $\mathcal{W}$) $S(s, \alpha)|_{\mathcal{S}_{tangible}} = S_\nu(s, \alpha)|_{\mathcal{S}_{tangible}}$ for all $\alpha \in Act$. This is a *vanishing representation with respect to* $\mathcal{W}$.
   - If no vanishing representation with respect to the current partition $\mathcal{W}$ can be found, s must be tangible with respect to $\mathcal{W}$. Add those states to $\mathcal{S}_{tangible}$.
   - Cross-check all other states if they also become tangible as an effect of s being tangible.
5. Find a new splitter (in the sense of [1]) with respect to the current partition and the current set of tangible states, i.e. a tuple $(C, \alpha, \mathcal{W})$ which indicates that class C needs to be refined w.r.t. a weak α transition.
6. Refine the partition according to the splitter and start next round.

The algorithm can be found in Alg. 1. By $DiracDet(s, \tau)$ we mean all distributions ν induced by $s \Rightarrow \nu$ by means of a Dirac determinate scheduler. It remains to define the ComputeInfo algorithm, FindWeakSplit algorithm and the Refine algorithm. As the Refine algorithm is standard, we omit it from this paper. The routine ComputeInfo just calculates the convex sets $S(s, \alpha)$ according to Remark 11. The routine FindWeakSplit given in Alg. 2 pretty much looks like the one given in [1] but it only "sees" tangible states that are provided as an additional parameter to the routine.

Algorithm 1 DecideWeakBisim

Require: Two MA as PA $P_1 = (\mathcal{S}_1, Act_1, \rightarrow_1, \emptyset, s_0)$, $P_2 = (\mathcal{S}_2, Act_2, \rightarrow_2, \emptyset, t_0)$

$\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$, $\mathcal{W} = \mathcal{S}$, $Act = Act_1 \cup Act_2$
for $s \in \mathcal{S}$, $\alpha \in Act$, $\nu \in DiracDet(s, \tau)$ **do**
  $S(s, \alpha) = ComputeInfo(s, \alpha)$
  $S_\nu(s, \alpha) = ComputeInfo(s, \alpha)$ on $(P_1 \cup P_2)_{(s,\nu)}$
end for
while $\mathcal{W}$ changes **do**
  $\mathcal{S}_{tangible} := \emptyset$
  while $\mathcal{S}_{tangible}$ changes **do**
    for $s \in \mathcal{S} \setminus \mathcal{S}_{tangible}$ **do**
      for $\nu \in DiracDet(s, \tau)$ where $\exists x \in Supp(\nu) : [x]_\mathcal{W} \neq [s]_\mathcal{W}$ **do**
        if $\forall \alpha \in Act : (S(s, \alpha)|_{\mathcal{S}_{tangible}})/\mathcal{W} = (S_\nu(s, \alpha)|_{\mathcal{S}_{tangible}})/\mathcal{W}$ **then**
          vanishing representation found **break**
        end if
      end for
      if no vanishing representation found **then**
        $\mathcal{S}_{tangible} := \mathcal{S}_{tangible} \cup \{s\}$
      end if
    end for
  end while
  $(C, \alpha, \mathcal{W}) = FindWeakSplit(\mathcal{S}_{tangible}, \mathcal{W}, \mathcal{S}, Act, S(\cdot, \cdot))$
  $\mathcal{W} = Refine(C, \alpha, \mathcal{W})$
end while
$P_1 \approx P_2$ iff $[s_0]_\mathcal{W} = [t_0]_\mathcal{W}$

Algorithm 2 FindWeakSplit (Find weak bisimulation splitter)

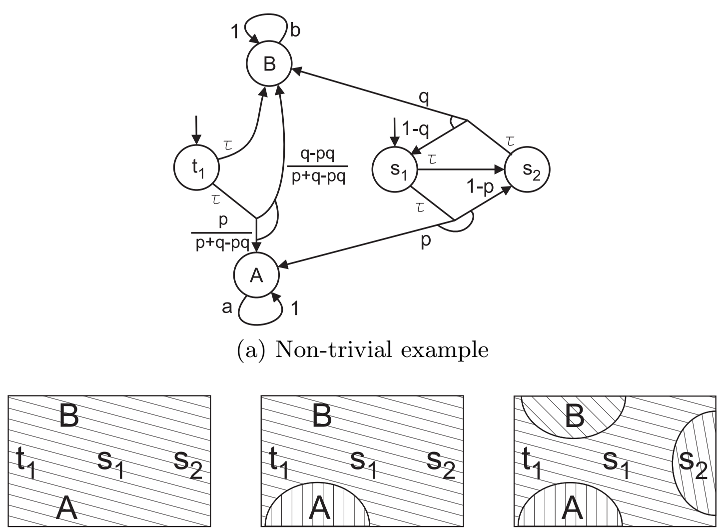
Require: Tangible states $\mathcal{S}_{tangible}$, partition $\mathcal{W}$, states $\mathcal{S}$, actions Act, Info $S(\cdot, \cdot)$

for $C_i \in \mathcal{W}$, $s, t \in C_i$, $\alpha \in Act$ **do**
  if $(S(s, \alpha)|_{\mathcal{S}_{tangible}})/\mathcal{W} \neq (S(t, \alpha)|_{\mathcal{S}_{tangible}})/\mathcal{W}$ **then**
    return $(C_i, \alpha, \mathcal{W})$
  end if
end for

Remark 13. The algorithm detects nn-vanishing states and finds their vanishing representations (regarding the last partition $\mathcal{W}$). Therefore we will be able to eliminate all nn-vanishing states and reach a form where only tangible states are present and where for every equivalence class only one state is used. This can be regarded as a kind of normal form.

5.1 Example

Suppose we are given the MA in Fig. 11a (there already transformed to a PA P) where $p, q \in (0, 1)$ (ignore for the moment the two initial states). This automaton can be seen as a condensed form of two different automata (starting with s_1 and t_1), where states A and B have been identified (to keep things short – if there were two copies of A and B: one for the left and one for the right automaton, they would be grouped during the progress of the algorithm). We want to show that $s_1 \approx_\Delta t_1$.



(a) Non-trivial example

(b) $\mathcal{W}_0$ (c) $\mathcal{W}_1$ (d) $\mathcal{W}_2$

Fig. 11: Example and partitions during algorithm run

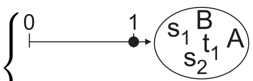
We assume that $p = q = \frac{1}{2}$, as the pictures are easier to draw in that case, but we would like to stress that the same arguments work for all other choices (as long as p and q are not equal to 0 or 1).

Remark 14. For didactical reasons we add dots for every result of a Dirac determinate scheduler (according to [1]) whenever we draw convex sets. Dots that are not extremal points may safely be omitted, as we are talking about convex sets.

First round: Start with the partition $\mathcal{W}_0 = \{\{s_1, s_2, t_1, A, B\}\}$ (cf. Fig. 11b). Observe that in the loop from line 9 to line 18 we can never find a vanishing representation, as no state may leave its equivalence class with some probability greater than zero. Therefore we get $\mathcal{S}_{tangible} = \{s_1, s_2, t_1, A, B\}$.

Now we have to find a splitter with respect to $(\mathcal{S}_{tangible}, \mathcal{W}_0)$. Suppose that we check the sets $S(\cdot, b)/\mathcal{W}_0$. Here we identify $s_1 \equiv s_2 \equiv t_1 \equiv A \equiv B$ and see that:

$S(x, b)/\mathcal{W}_0 = \Big\{$



for $x \in \{B, s_1, s_2, t_1\}$

$\emptyset$ otherwise

So we have found a splitter. Refining according to $(\{s_1, s_2, t_1, A, B\}, b, \mathcal{W}_0)$ leads to $\mathcal{W}_1 = \{\{s_1, s_2, t_1, B\}, \{A\}\}$ (cf. Fig. 11c).

Second round: We first have to detect the nn-vanishing states with respect to the current partition. We calculate $S(x, \tau)$ for every state, verify if it is possible to reach another equivalence class and see whether one single τ transition suffices. The values of $S(x, \tau)$ are given in Tab. 1. Firstly notice that both A and B cannot

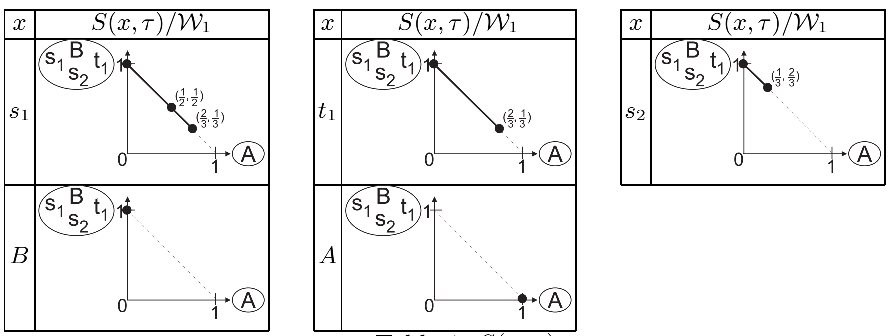


Table 1: $S(x, \tau)$

be nn-vanishing, as they have no possibility of leaving their equivalence classes. Notice also, that even if s_2 is vanishing, as it has only one single emanating τ transition, we cannot detect it as nn-vanishing (the only van. rep. that leaves the class would be $P_{(s_2, \frac{1}{3}\Delta_A \oplus \frac{2}{3}\Delta_B)}$, but $S(s_2, b)/\mathcal{W}_1 \neq S_{\frac{1}{3}\Delta_A \oplus \frac{2}{3}\Delta_B}(s_2, b)/\mathcal{W}_1$). Regarding s_1 we see that we cannot omit transition $s_1 \xrightarrow{\tau} \frac{1}{2}\Delta_A \oplus \frac{1}{2}\Delta_{s_2}$, as $S_{(\Delta_{s_2})}(s_1, \tau)/\mathcal{W}_1 = S(B, \tau)/\mathcal{W}_1 \neq S(s_1, \tau)$. But notice also that $s_1 \xrightarrow{\tau} \Delta_{s_2}$ cannot be omitted, as $S(s_1, b)/\mathcal{W}_1 = S(B, \tau)/\mathcal{W}_1$, but $S_{(\frac{1}{2}\Delta_A \oplus \frac{1}{2}\Delta_{s_2})}(s_1, b)/\mathcal{W}_1 = \emptyset$. So we see that s_1 cannot be nn-vanishing. With the same argument we see that also t_1 cannot be nn-vanishing. Therefore we get $\mathcal{S}_{tangible} = \{s_1, s_2, t_1, A, B\}$.

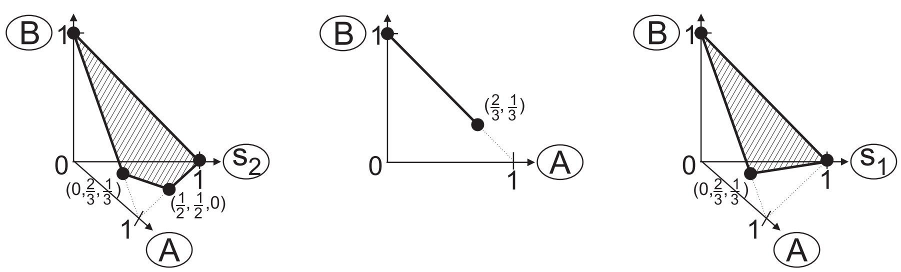
Now we look for splitters with respect to $(\mathcal{W}_1, \mathcal{S}_{tangible})$. Looking at $C = \{s_1, s_2, t_1, B\}$ we see in routine FindWeakSplit that we can use a splitter $(C, \tau, \mathcal{W}_1)$ and get the partition $\mathcal{W}_2 = \{\{s_1, t_1\}, \{s_2\}, \{A\}, \{B\}\}$ (cf. Fig. 11d, note that $S(s_1, \tau)/\mathcal{W}_1 = S(t_1, \tau)/\mathcal{W}_1$, as $(\frac{1}{2}, \frac{1}{2})$ is *not* a generator of the convex set).

Third round: We first have to detect nn-vanishing states. It is clear that s_2 must be nn-vanishing as it can leave its class and only has a single outgoing τ transition. With the same arguments as above we see that both s_1 and t_1 must be tangible. So we get $\mathcal{S}_{tangible} = \{s_1, t_1, A, B\}$.

Now we again can look for splitters, but have to consider the restriction to $\mathcal{S}_{tangible}$. Notice that with coordinates $[s_1] = [t_1]$, $[s_2]$, $[A]$, $[B]$ we have

$$S(s_1, \tau)/\mathcal{W}_2 = CHull(\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\\frac{1}{2}\\\frac{1}{2}\\0\end{pmatrix}, \begin{pmatrix}0\\0\\\frac{2}{3}\\\frac{1}{3}\end{pmatrix}).$$

We want to calculate the restriction $S(s_1, \tau)|_{\mathcal{S}_{tangible}}/\mathcal{W}_2$. Let us for the moment ignore the vertex $[s_1] \in S(s_1, \tau)/\mathcal{W}_2$. Then we get the picture in Fig. 12a for $S(s_1, \tau)|_{s_1=0}/\mathcal{W}_2$. We see that the restriction of this set to $\mathcal{S}_{tangible}$ gives



(a) $S(s_1, \tau)|_{s_1=0}/\mathcal{W}_2$ (b) $S(s_1, \tau)|_{s_1, s_2=0}/\mathcal{W}_2$ (c) $S(s_1, \tau)|_{\mathcal{S}_{tangible}}/\mathcal{W}_2$

Fig. 12: Convex sets

only the line from $(0, 0, 1)$ to $(0, \frac{2}{3}, \frac{1}{3})$ (cf. Fig. 12b), therefore we conclude that $S(s_1, \tau)|_{\mathcal{S}_{tangible}}/\mathcal{W}_2$ is the set given in Fig. 12c. We get the same set for $S(t_1, \tau)|_{\mathcal{S}_{tangible}}/\mathcal{W}_2$ (here, no nn-vanishing state has to be ignored). Looking at all other sets $S(\cdot, \cdot)$ we find no other splitter, so $\mathcal{W}_2$ cannot be refined.

With the partition $\mathcal{W}_2$ and the set of stable states $\mathcal{S}_{tangible}$ we have reached our fixed point, the algorithm terminates and we see that s_1 and t_1 are still in the same partition, so they are weakly bisimilar.

Remark 15 (Optimisations). A few optimisations can be performed:

– In every round states without other τ transitions than the loop will always be detected as stable, so this set can be separated as a preprocessing step.
– All states with only one single outgoing τ transitions and without non-τ transitions can safely be eliminated in advance (cf. Lemma 3).

6 Conclusion

We have shown that weak and naïve weak bisimulation for MA are closely related by an appropriate formulation of elimination and that the two notations coincide, when no non-naïvely vanishing states are present. We have presented an algorithm for deciding weak MA bisimilarity that, as a by-product, finds non-naïvely vanishing states and their corresponding vanishing representations. This can also be used to define normal forms for MA. Even with the magnificent results of [5] it remains an open question whether weak MA bisimulation can be decided in polynomial time.

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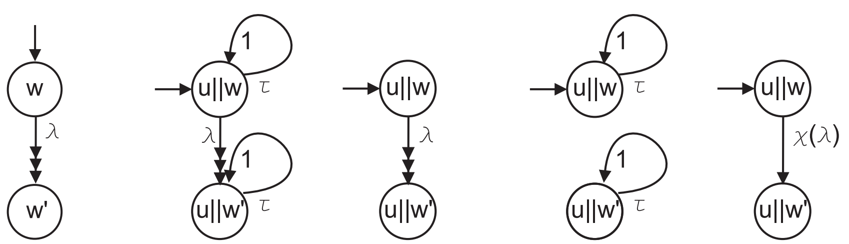
A Timed actions with rate zero

The following remark shows why we modified the definition of P_s (see Def. 4, for the original definiton see [2,3,4]). To explain the argument, we need to recall the definition of parallel composition of MA (cf. Definition 2 in [3]). As in our example we only use the unsynchronised case, we only note that a state in the composed MA may perform all transitions of its subcomponents. As usual, we denote parallel composition by the || operator.



(a) M_4 (b) $M_1||M_4$ (c) $M_2||M_4$ (d) $\mathcal{PA}(M_1||M_4)$ (e) $\mathcal{PA}(M_2||M_4)$

Fig. 13: Counterexample to Theorem 3 in [3,4] (unless $\chi(0)$ is defined)

Remark 16 (Parallel composition and $\chi(0)$*).* We discuss briefly the case when a stable state has no outgoing timed transition. Note that the case $rate(u) = 0$ remains undefined in [2,3,4] (division by zero), therefore no $\chi(0)$ action can be generated there. This would be problematic, as without the $\chi(0)$ action we would have $\mathcal{PA}(M_1) \approx \mathcal{PA}(M_2)$ in Fig. 1, but parallel composition with the M_4 from Fig. 13a would lead to different results: $M_1||M_4$ is given in Fig. 13b, $M_2||M_4$ is given in Fig. 13c. The corresponding PA are given in Fig. 13d and Fig. 13e, respectively. Note that the results are different modulo weak bisimulation and thus falsifying Theorem 3 in [3,4] which claims that weak bisimulation is a congruence with respect to parallel composition. So we would like to stress that it is necessary to use Def. 4 for the successor probability and not the definitions from [2,3,4]. Only with this definition Theorem 3 in [3,4] can hold. We conjecture that the results of [3,4,2] still hold using Def. 4, but we did not prove that. We only have shown by the counterexample that without considering $\chi(0)$, the results cannot hold.

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