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Numerische Simulation auf massiv parallelen Rechnern

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A note on anisotropic interpolation error estimates for isoparametric quadrilateral finite elements

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Abstract. Anisotropic local interpolation error estimates are derived for quadrilateral and hexahedral Lagrangian finite elements with straight edges. These elements are allowed to have diameters with different asymptotic behaviour in different space directions. The case of affine elements (parallelepipeds) with arbitrarily high degree of the shape functions is considered first. Then, a careful examination of the multi-linear map leads to estimates for certain classes of more general, isoparametric elements. As an application, the Galerkin finite element method for a reaction diffusion problem in a polygonal domain is considered. The boundary layers are resolved using anisotropic trapezoidal elements.

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1 Introduction

The classical finite element approximation theory relies on the condition that the elements are isotropic, that means, the lengths of all sides of the element are of the same order and their interior angles do not degenerate, see for example [7, 9]. However, in recent years elements were successfully applied which violate these conditions; they are called *anisotropic*. The diameters of such elements have different asymptotic behaviour in different spatial directions. Applications include the approximation of edge singularities in diffusion dominated problems [1], of boundary and interior layers [4, 3, 16], or simply the meshing of narrow domains like the gap between rotor and stator in an electrical machine.

First attempts to treat such elements were made in several papers including [5, 10, 13, 14, 23, 25] by proving local interpolation error estimates where only the *largest* diameter appears in the result. So these authors did not derive the possible advantage of using elements with different diameters in different directions.

This remedy was removed in [1, 3, 17, 19, 24] by proving various sharper (*anisotropic*) interpolation error estimates for simplicial and cuboidal elements in two and three dimensions. However, up to now the theory for quadrilateral and hexahedral elements is limited to tensor product elements (rectangles and bricks). Parallelepipeds and isoparametric elements were not considered yet. But such elements are of importance if quadrilateral finite element meshes are investigated for arbitrary polygonal domains. They are focused in the present paper.

The outline is as follows: For clarity we restrict ourselves first to two dimensions. After introducing some notation and discussing the simple case of parallelograms we elaborate in Section 3 the bilinear transformation and derive anisotropic interpolation error estimates for quadrilateral elements with straight edges. In Section 4 we will see that some but not all of these results extend to three dimensions.

In a final section we sketch an application of these results and derive an optimal finite element error estimate for a reaction diffusion problem in a general polygonal domain where the boundary layer is resolved using anisotropic trapezoidal elements. We point out that this error estimate cannot be obtained using previous interpolation results, see Remark 3 on page 13. Note that our application consists in an a-priori error estimate. For first attempts to construct adaptive methods using anisotropic elements we refer to [11, 12, 15, 20, 22].

2 Notation and results for affine elements

Consider isoparametric quadrilateral elements $e \in I\!R^2$ with, for simplicity, straight edges. Introduce the reference element $\hat{e} = (0,1)^2$ and denote as in [7, Section 2.2] by Q_k the space of all polynomials of the form $q(\hat{x}) = \sum_{0\le\alpha_1,\alpha_2\le k} c_\alpha \hat{x}^\alpha$, $\hat{x} = (\hat{x}_1, \hat{x}_2)$. Throughout the paper, the parameter k will characterize the polynomials in this sense. We use a multi-index notation with

$$\alpha := (\alpha_1,\alpha_2),\ \ |\alpha| := \alpha_1+\alpha_2,\ \ m\alpha := (m\alpha_1, m\alpha_2),\ \ \hat{x}^\alpha := \hat{x}_1^{\alpha_1}\hat{x}_2^{\alpha_2},\ \ \hat{D}^\alpha := \frac{\partial^{|\alpha|}}{\partial\hat{x}_1^{\alpha_1}\partial\hat{x}_2^{\alpha_2}},$$

$\alpha_1, \alpha_2 \in I\!N_0$, $m \in I\!R^+$.

The shape functions $\hat{\psi}_1 := (1-\hat{x}_1)(1-\hat{x}_2)$, $\hat{\psi}_2 := \hat{x}_1(1-\hat{x}_2)$, $\hat{\psi}_3 := \hat{x}_1\hat{x}_2$, $\hat{\psi}_4 := (1-\hat{x}_1)\hat{x}_2$ in the bilinear case are also used for the mapping $x = F(\hat{x})$ of $\hat{e}$ onto e: Let $X^{(i)} = (X_1^{(i)}, X_2^{(i)})^T$, $i = 1, \ldots, 4$, denote the vertices of e, then

$$x = F(\hat{x}) := \sum_{i=1}^{4} X^{(i)}\hat{\psi}_i(\hat{x}) \in (Q_1)^2.$$

We assume that e is convex, then this mapping is invertible [9, p. 105]. Note that $X^{(i)}$, F, and several other identifiers below depend on e, but we omit another index to keep the notation short.

Consider now the general case $k \geq 1$. Denote by $\hat{\varphi}_i(\hat{x})$, $i = 1, \ldots, (k+1)^2$, the usual nodal shape functions corresponding to the set $\{0, \frac{1}{k}, \frac{2}{k}, \ldots, \frac{k-1}{k}, 1\}^2$ of nodal points. Then we define via $\varphi_i(x) := \hat{\varphi}_i(F^{-1}(x))$ the ansatz functions on e. Note that, in contrast to affine elements, these functions are not polynomial in general. In the special case of e being a parallelogram, the transformation $F(\hat{x})$ is affine ($X^{(1)} - X^{(2)} + X^{(3)} - X^{(4)} = 0$).

Let $I^{(k)}$ be the Lagrangian interpolation operator on the reference element $\hat{e}$. The interpolation operator on e is then defined by $(I_h^{(k)} v)(x) := I^{(k)}\hat{v}(\hat{x})$, where $\hat{v}(\hat{x}) := v(F(\hat{x}))$.

Finally, let $W^{m,p}(e)$, $m \in I\!N_0$, $p \in [1, \infty]$, be the usual Sobolev spaces with the norm and the special seminorm

$$\|v; W^{m,p}(e)\|^p := \sum_{|\alpha| \leq m} \int_e |D^\alpha v|^p dx, \quad |v; W^{m,p}(e)|^p := \sum_{|\alpha| = m} \int_e |D^\alpha v|^p dx,$$

and the usual modification for $p = \infty$. In general, we will write $L_p(e)$ for $W^{0,p}(e)$. The symbol C is used for a generic positive constant which may be of different value at each occurrence. But C is always independent of the element e, in particular of its size, and of the function under consideration.

To summarize interpolation error estimates on the reference element $\hat{e}$ which are suited for anisotropic elements e, we formulate the following theorem, see [8, Lemma 5] and [1, Theorems 3 and 4]. Estimate (2) was proved in [18, 25] for $k = 1$, $p = 2$, and in [10] for $k = 1$, $p > 2$, and for $k \geq 2$, $p \geq 1$, as well.

Theorem 1 *Assume $\hat{v} \in W^{k+1,p}(\hat{e})$, $1 \leq p \leq \infty$, and let γ be a multi-index with $|\gamma| = 1$. Then the estimates*

$$\begin{aligned} \|\hat{v} - I^{(k)}\hat{v}; L_p(\hat{e})\| &\leq C \sum_{|\alpha|=1} \|\hat{D}^{(k+1)\alpha}\hat{v}; L_p(\hat{e})\|, && (1) \\ \|\hat{D}^\gamma(\hat{v} - I^{(k)}\hat{v}); L_p(\hat{e})\| &\leq C|\hat{D}^\gamma\hat{v}; W^{k,p}(\hat{e})| && (2) \end{aligned}$$

hold. If $\hat{v} \in W^{k+2,p}(\hat{e})$, $1 \leq p \leq \infty$, then we have also

$$\|\hat{D}^\gamma(\hat{v} - I^{(k)}\hat{v}); L_p(\hat{e})\| \leq C\|\hat{D}^{(k+1)\gamma}\hat{v}; L_p(\hat{e})\| + C|\hat{D}^\gamma\hat{v}; W^{k+1,p}(\hat{e})|. \quad (3)$$

The proof of anisotropic interpolation error estimates reduces now to the transformation of the estimates in Theorem 1 to the quadrilateral e. The simplest case of e being a rectangle, where $F(\hat{x}) = (h_1\hat{x}_1, h_2\hat{x}_2)^T + X^{(1)}$, was considered in [1]. In a first step we will generalize this to parallelograms which satisfy the following two conditions, compare Figure 1.

Interior angle condition: The interior angles γ_i of the element e are bounded by $0 < \gamma_* \leq \gamma_i \leq \pi - \gamma_*$, $i = 1, \ldots, 4$, where the constant γ_* is independent of e, in particular of the mesh size.

Coordinate system condition: The angle ψ between the longest side of the element e and the x_1-axis is bounded by $|\sin\psi| \leq Ch_2/h_1$.

Here, h_1 denotes the length of the longest edge of e and $h_2 := \mathrm{meas}_2(e)/h_1$ is the corresponding height.

We point out that the coordinate system condition is not as restrictive as it might look. The x_1, x_2-coordinate system can be fitted to the boundary or some other manifold where

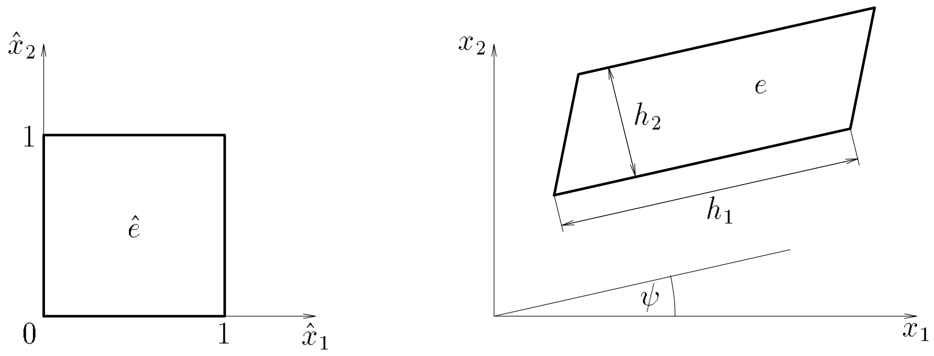

Figure 1: Illustration of the affine element.

peculiarities in the solution arise. It is only demanded that this system is independent of the single element e which is considered here. So we could also set $\psi = 0$. On the other hand, the introduction of this condition prevents discussions about whether the direction of the longest edge or of the longest diagonal is the stretching direction.

We recall that the transformation can be realized by

$$x = F(\hat{x}) = B\hat{x} + b \tag{4}$$

with $B = (b_{ij})_{i,j=1}^2 \in I\!R^{2\times 2}$, $b \in I\!R^2$, and

$$|\det D| = |\det B| = h_1 h_2, \tag{5}$$

where D is the Jacobi matrix of this transformation. Because this situation corresponds completely to the case of triangular elements we have the following estimates for the entries of the matrices B and $B^{-1} = (b_{ij}^{(-1)})_{i,j=1}^2$, see [2, Theorem A4].

$$\begin{aligned} |b_{ij}| &\leq C\min\{h_i, h_j\}, \quad i,j = 1,2, & (6)\\ |b_{ij}^{(-1)}| &\leq C\min\{h_i^{-1}, h_j^{-1}\}, \quad i,j = 1,2. & (7) \end{aligned}$$

From this we conclude by simple transformation rules the following estimates for the transformation of the derivatives,

$$|D^\gamma v| \leq C \sum_{|\beta|=|\gamma|} h^{-\beta} |\hat{D}^\beta \hat{v}|, \quad |\hat{D}^\beta \hat{v}| \leq C h^\beta \sum_{|t|=|\beta|} |D^t v|, \quad |\hat{D}^\alpha \hat{v}| \leq C \sum_{|s|=|\alpha|} h^s |D^s v|, \tag{8}$$

$h^\alpha := h_1^{\alpha_1} h_2^{\alpha_2}$. We can now imply the anisotropic interpolation error estimates corresponding to Theorem 1.

Theorem 2 *Assume that e is a parallelogram which satisfies the interior angle condition and the coordinate system condition. Let γ be a multi-index with $|\gamma| = 1$ and $v \in W^{k+1,p}(e)$, $1 \leq p \leq \infty$. Then the estimates*

$$\begin{aligned} \|v - I_h^{(k)} v; L_p(e)\|^p &\leq C \sum_{|\alpha|=k+1} h^{\alpha p} \|D^\alpha v; L_p(e)\|^p, & (9)\\ \|D^\gamma (v - I_h^{(k)} v); L_p(e)\|^p &\leq C \sum_{|\alpha|=k} h^{\alpha p} |D^\alpha v; W^{1,p}(e)|^p & (10) \end{aligned}$$

hold.

Note that the transformation of (3) makes sense only in the case of rectangular elements where mixed derivatives of order $k+1$ can be avoided,

$$\|D^\gamma(v - I_h^{(k)}v); L_p(e)\| \le Ch^{k\gamma}\|D^{(k+1)\gamma}; L_p(e)\| + C\sum_{|\alpha|=k+1} h^{\alpha p}\|D^{\alpha+\gamma}v; L_p(e)\|.$$

Proof We give the proof for the slightly more difficult case (10); the proof of (9) is similar. By (8) and Theorem 1 we have

$$\begin{aligned}
\|D^\gamma(v - I_h^{(k)}v); L_p(e)\|^p &\le Ch_1h_2\sum_{|\beta|=1} h^{-\beta p}\|\hat{D}^\beta(\hat{v} - I^{(k)}\hat{v}); L_p(\hat{e})\|^p \\
&\le Ch_1h_2\sum_{|\alpha|=k}\sum_{|\beta|=1} h^{-\beta p}\|\hat{D}^{\alpha+\beta}\hat{v}; L_p(\hat{e})\|^p \\
&\le C\sum_{|\alpha|=k}\sum_{|\beta|=1} h^{-\beta p}\sum_{|t|=1}\sum_{|s|=k} h^{\beta p}h^{sp}\|D^{s+t}v; L_p(e)\|^p \\
&= C\sum_{|t|=1}\sum_{|s|=k} h^{sp}\|D^{s+t}v; L_p(e)\|^p,
\end{aligned}$$

and the theorem is proved. □

3 Bilinear isoparametric elements

We consider the isoparametric transformation as a perturbation of an affine transformation. Let $\tilde{e}$ be a rectangular element with edges parallel to the axes of the coordinate system. The coordinates of the vertices of $\tilde{e}$ are $X^{(i)}$, $i = 1, \dots, 4$. The isoparametric element e is a perturbation of $\tilde{e}$, the coordinates of its vertices are $X^{(i)} + a^{(i)}$, $i = 1, \dots, 4$. Denote by

$$\begin{aligned}
\tilde{F}(\hat{x}) &= X^{(1)} + B\hat{x}, \quad B = \mathrm{diag}(h_1, h_2), \\
F(\hat{x}) &= \tilde{F}(\hat{x}) + \sum_{i=1}^4 a^{(i)}\hat{\psi}_i(\hat{x}),
\end{aligned}$$

the transformation of $\hat{e}$ to $\tilde{e}$ and e, respectively, that means $\tilde{e} = \tilde{F}(\hat{e})$, $e = F(\hat{e})$.

The Jacobi matrix of the transformation F is

$$D = D(\hat{x}) = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} = B + \sum_{i=1}^4 \begin{pmatrix} a_1^{(i)}\dfrac{\partial\hat{\psi}_i}{\partial\hat{x}_1} & a_1^{(i)}\dfrac{\partial\hat{\psi}_i}{\partial\hat{x}_2} \\ a_2^{(i)}\dfrac{\partial\hat{\psi}_i}{\partial\hat{x}_1} & a_2^{(i)}\dfrac{\partial\hat{\psi}_i}{\partial\hat{x}_2} \end{pmatrix}.$$

In order to keep properties like (5)–(7) we demand the existence of a_0 and $a = (a_1, a_2)$ with

$$|a_i^{(j)}| \le a_ih_2, \quad 0 \le a_i \le C, \quad i = 1, 2, \ j = 1, \dots, 4, \tag{11}$$

$$\frac{1}{2} - \frac{h_2}{h_1}a_1 - a_2 \ge a_0 > 0. \tag{12}$$

While a_0 is a constant, the numbers a_i are allowed to depend on h_1 and h_2, within the limitations given by (11), (12).

Lemma 3 *The conditions (11), (12) imply for all $\hat{x} \in \hat{e}$ the estimates*

$$\begin{aligned}
C_1h_1h_2 \ \le \ |\det D(\hat{x})| &\le C_2h_1h_2 &(13)\\
|d_{ij}(\hat{x})| &\le C\min\{h_i, h_j\}, \quad i, j = 1, 2, &(14)\\
|d_{ij}^{(-1)}(\hat{x})| &\le C\min\{h_i^{-1}, h_j^{-1}\}, \quad i, j = 1, 2, &(15)
\end{aligned}$$

where $d_{ij}^{(-1)}$ are the entries of the inverse of the Jacobi matrix D.

Proof By the calculation of $\frac{\partial \hat{\psi}_i}{\partial \hat{x}_j}$ we obtain with (11) and (12)

$$|d_{11} - h_1| = \left|(1-\hat{x}_2)(a_1^{(2)} - a_1^{(1)}) + \hat{x}_2(a_1^{(3)} - a_1^{(4)})\right| \;\leq\; 2a_1h_2$$

and similarly $|d_{12}| \leq 2a_1h_2$, $|d_{21}| \leq 2a_2h_2$, and $(1-2a_2)h_2 \leq d_{22} \leq (1+2a_2)h_2$. Consequently,

$$\begin{aligned}
\det J &= d_{11}d_{22} - d_{12}d_{21} \;\geq\; (h_1 - 2a_1h_2)(1-2a_2)h_2 - 4a_1a_2h_2^2 \\
&= h_1h_2(1 - 2\tfrac{h_2}{h_1}a_1 - 2a_2) \;\geq\; 2a_0h_1h_2, \\
\det J &\leq (1 + 2\tfrac{h_2}{h_1}a_1)h_1(1+2a_2)h_2 + 4a_1a_2h_2^2 \;\leq\; Ch_1h_2,
\end{aligned}$$

and (13) and (14) are proved. The estimate (15) is a direct consequence using the explicit representation of the inverse. □

Remark 1 Note that there is virtually no restriction on a_1 if $h_2 \ll h_1$. Note further that the condition to a_2 could be weakened if the numbers $a_2^{(i)}$, $i = 1,\ldots,4$, satisfy $\operatorname{sign} a_2^{(1)} = \operatorname{sign} a_2^{(4)}$ and $\operatorname{sign} a_2^{(2)} = \operatorname{sign} a_2^{(3)}$. This is the reason why the affine elements from Section 2 do satisfy (11) but with constants not necessarily satisfying (12). As another alternative we could consider perturbations of parallelograms $\tilde{e}$ satisfying the conditions of Section 2. The following results would remain true but the angle ψ would have to be involved in (12). We chose a rectangle to keep our explanations as clear as possible.

Because for the second order derivatives of the transformation F the relations

$$\frac{\partial^2 x_i}{\partial \hat{x}_j^2} = 0, \quad i,j = 1,2, \tag{16}$$

hold, we conclude by analogy to (8) for pure (non-mixed) derivatives $D^\alpha v$ with $\alpha = k\gamma$ ($k \in I\!N$, $|\gamma| = 1$)

$$|\hat{D}^{k\gamma}\hat{v}| \leq C \sum_{|s|=k} h^s |D^s v|. \tag{17}$$

Using (1) we obtain immediately that the anisotropic interpolation error estimate (9) holds in the isoparametric case as well.

The drawback for estimates of the derivatives of the interpolation error is that mixed derivatives appear at the right hand side of (2) and (3). In view of

$$\frac{\partial^2 x_i}{\partial \hat{x}_1 \partial \hat{x}_2} = a_i^{(1)} - a_i^{(2)} + a_i^{(3)} - a_i^{(4)}, \qquad \left|\frac{\partial^2 x_i}{\partial \hat{x}_1 \partial \hat{x}_2}\right| \leq 4a_ih_2, \quad i = 1,2, \tag{18}$$

this implies that in the transformation of the k-th order derivative $\hat{D}^\alpha$ also derivatives D^β of order $\left[\frac{k+1}{2}\right], \ldots, k$ will appear. Here, $[z]$ defines the largest integer which is less or equal z. Therefore, the anisotropic interpolation error estimate will not be of the quality of (10). We obtain the following result.

Theorem 4 *Consider a rectangular element $\tilde{e}$ with sides of length h_1 and h_2, $h_1 \geq h_2$, which are parallel to the axes of the x_1, x_2-coordinate system. The coordinates of the four vertices are perturbed by vectors $a^{(i)} = (a_1^{(i)}, a_2^{(i)})^T$ satisfying at least (11), (12). The resulting element* *is denoted by e. Then for $k \in I\!N$, $v \in W^{k+1,p}(e)$, $1 \le p \le \infty$, the following anisotropic interpolation error estimates hold:*

$$\|v - I_h^{(k)} v; L_p(e)\|^p \le C \sum_{|\alpha|=k+1} h^{\alpha p} \|D^\alpha v; L_p(e)\|^p, \tag{19}$$

$$\begin{aligned} |v - I_h^{(k)} v; W^{1,p}(e)|^p \le\ & C \sum_{|\alpha|=k} h^{\alpha p} |D^\alpha v; W^{1,p}(e)|^p + \\ & + C \sum_{r=[k/2]+1}^{k} h_2^{(k-r)p} \sum_{|\alpha|=2r-k-1} \sum_{|\beta|=k+1-r} h^{\alpha p} a^{\beta p} \|D^{\alpha+\beta} v; L_p(e)\|^p. \end{aligned} \tag{20}$$

If even $v \in W^{k+2,p}(e)$, $1 \le p \le \infty$, then

$$\begin{aligned} |v - I_h^{(k)} v; W^{1,p}(e)|^p \le\ & C \sum_{k \le |\alpha| \le k+1} h^{\alpha p} |D^\alpha v; W^{1,p}(e)|^p + \\ & + C \sum_{r=[(k+1)/2]+1}^{k+1} h_2^{(k+1-r)p} \sum_{|\alpha|=2r-k-2} \sum_{|\beta|=k+2-r} h^{\alpha p} a^{\beta p} \|D^{\alpha+\beta} v; L_p(e)\|^p. \end{aligned} \tag{21}$$

Proof The validity of (19) was already discussed above. For the other estimates we have to transform mixed derivatives and start with a transformation formula in tensor form, see [6, Relations (2.9)–(2.10)]:

$$\begin{aligned} \hat{D}^m \hat{v} \ :=\ (D^\alpha)_{|\alpha|=m} &= \sum_{r=1}^{m} D^r v \sum_{\underline{i} \in E(m,r)} c_{\underline{i}} \prod_{q=1}^{m} (D^q F)^{i_q}, \\ E(m,r) &:= \left\{ \underline{i} \in I\!N_0^m : \sum_{q=1}^{m} i_q = r, \sum_{q=1}^{m} q\, i_q = m \right\}. \end{aligned}$$

Because third derivatives of F vanish in our case it suffices to consider the set

$$\begin{aligned} E(m,r) &= \{(i_1, i_2) \in I\!N_0^2 : i_1 + i_2 = r, i_1 + 2 i_2 = m\} \\ &= \{(i_1, i_2) \in I\!N_0^2 : i_1 = 2r - m, i_2 = m - r\} \end{aligned} \tag{22}$$

(let $i_3 = \ldots = i_m = 0$) which yields $r \ge [(m+1)/2]$ and

$$\hat{D}^m \hat{v} = \sum_{r=[(m+1)/2]}^{m} c_r\, D^r v\, (\hat{D}^1 F)^{2r-m} (\hat{D}^2 F)^{m-r}.$$

Now we extract single derivatives from this relation. For this we split multi-indices in the form $\alpha = \sum_{i=1}^{|\alpha|} \alpha^{(i)}$ with $|\alpha^{(i)}| = 1$, $i = 1, \ldots, |\alpha|$. We obtain with $|\alpha| = m$

$$\begin{aligned} \hat{D}^\alpha \hat{v} &= \sum_{r=[(m+1)/2]}^{m} c_r \sum_{|s|=r} D^s v \left(\prod_{i=1}^{2r-m} \hat{D}^{\alpha^{(i)}} x^{s^{(i)}} \right) \left(\prod_{j=1}^{m-r} \hat{D}^{\alpha^{(2r-m+2j-1)} + \alpha^{(2r-m+2j)}} x^{s^{(2r-m+j)}} \right), \\ |\hat{D}^\alpha \hat{v}| &\le C \sum_{r=[(m+1)/2]}^{m} \sum_{|s|=r} |D^s v| \left(\prod_{i=1}^{2r-m} \min\left\{ h^{\alpha^{(i)}}; h^{s^{(i)}} \right\} \right) \left(\prod_{j=1}^{m-r} a^{s^{(2r-m+j)}} \right) h_2^{m-r} \\ &= C \sum_{|s|=m} |D^s v| \prod_{i=1}^{m} \min\left\{ h^{\alpha^{(i)}}; h^{s^{(i)}} \right\} + C \sum_{r=[(m+1)/2]}^{m-1} \sum_{|s|=2r-m} \sum_{|t|=m-r} h^s a^t h_2^{m-r} |D^{s+t} v|. \end{aligned}$$

Note that in view of (16) some terms at the right hand side could be omitted but the quality of the following statements remains.

Consider now the transformation of (2). Set $m = k+1$ in the formula above, then we get by analogy to the proof of Theorem 2 for $|\gamma| = 1$:

$$\begin{aligned}
\|D^\gamma(v - I_h^{(k)}v); L_p(e)\|^p &\le Ch_1h_2 \sum_{|\beta|=1} h^{-\beta p}\|\hat{D}^\beta(\hat{v} - I^{(k)}\hat{v}); L_p(\hat{e})\|^p \\
&\le Ch_1h_2 \sum_{|\alpha|=k}\sum_{|\beta|=1} h^{-\beta p}\|\hat{D}^{\alpha+\beta}\hat{v}; L_p(\hat{e})\|^p \\
&\le C\sum_{|\alpha|=k}\sum_{|\beta|=1} h^{-\beta p}\Bigg(h^{\beta p}\sum_{|s|=k}\sum_{|t|=1} h^{sp}\|D^{s+t}v; L_p(e)\|^p + \\
&\qquad + h_2^p \sum_{r=[k/2]+1}^{k} h_2^{(k-r)p}\sum_{|s|=2r-k-1}\sum_{|t|=k+1-r} h^{sp}a^{tp}\|D^{s+t}v; L_p(e)\|^p\Bigg) \\
&\le C\sum_{|s|=k}\sum_{|t|=1} h^{sp}\|D^{s+t}v; L_p(e)\|^p + C\sum_{r=[k/2]+1}^{k} h_2^{(k-r)p}\sum_{|s|=2r-k-1}\sum_{|t|=k+1-r} h^{sp}a^{tp}\|D^{s+t}v; L_p(e)\|^p.
\end{aligned}$$

Thus (20) is proved. The remaining estimate is obtained by analogy using (3) and the transformation formula (17) for pure derivatives. □

Let us focus now some special cases. For $k = 1$ the estimate (20) means that the approximation order is not better than $\max\{a_1, a_2\}$. For the particular case

$$a_i \le Ch_i, \quad i = 1, 2, \tag{23}$$

we obtain

$$|v - I_h^{(1)}v; W^{1,p}(e)|^p \le C\sum_{|\alpha|=1} h^{\alpha p}\|D^\alpha v; W^{1,p}(e)\|^p.$$

For quadratic ansatz functions we get from (20)

$$|v - I_h^{(2)}v; W^{1,p}(e)|^p \le C\begin{cases} \sum\limits_{1\le|\alpha|\le 2} h^{\alpha p}|D^\alpha v; W^{1,p}(e)|^p & \text{for } a \text{ satisfying (12)}, \\ \sum\limits_{|\alpha|=2} h^{\alpha p}\|D^\alpha v; W^{1,p}(e)\|^p & \text{for } a \text{ satisfying (23)}. \end{cases} \tag{24}$$

In each case we have both second and third order derivatives at the right hand side. Note that these second order derivatives as well as the first order derivatives in the case $k = 1$ can be omitted for isotropic elements. This is based on an estimate of type (1) on the reference element, where no mixed derivatives appear. But such an estimate is not applicable for anisotropic elements. One would get terms of the order $h_2^{-1}h_1^{k+1}$ at the right hand side.

The stronger assumption $v \in W^{k+2,p}(e)$ leads for $k = 1$ to

$$|v - I_h^{(1)}v; W^{1,p}(e)|^p \le C\sum_{1\le|\alpha|\le 2} h^{\alpha p}|D^\alpha v; W^{1,p}(e)|^p \quad \text{for } a \text{ satisfying (12)}.$$

We do not get an improvement with (23) instead of (12). For $k = 2$, however, estimate (21) gives only marginal advantage in comparison with (24). We get even fourth order derivatives at the right hand side but the second order terms remain:

$$|v - I_h^{(2)}v; W^{1,p}(e)|^p \;\le\; C\begin{cases} \sum\limits_{2\le|\alpha|\le 3} h^{\alpha p}|D^\alpha v; W^{1,p}(e)|^p + h_2^p|v; W^{2,p}(e)|^p \\ \hfill \text{for } a \text{ satisfying (12)}, \\ \sum\limits_{2\le|\alpha|\le 3} h^{\alpha p}|D^\alpha v; W^{1,p}(e)|^p + h_2^p\sum\limits_{|\alpha|=2} h^{\alpha p}\|D^\alpha v; L_p(e)\|^p \\ \hfill \text{for } a \text{ satisfying (23)}. \end{cases}$$

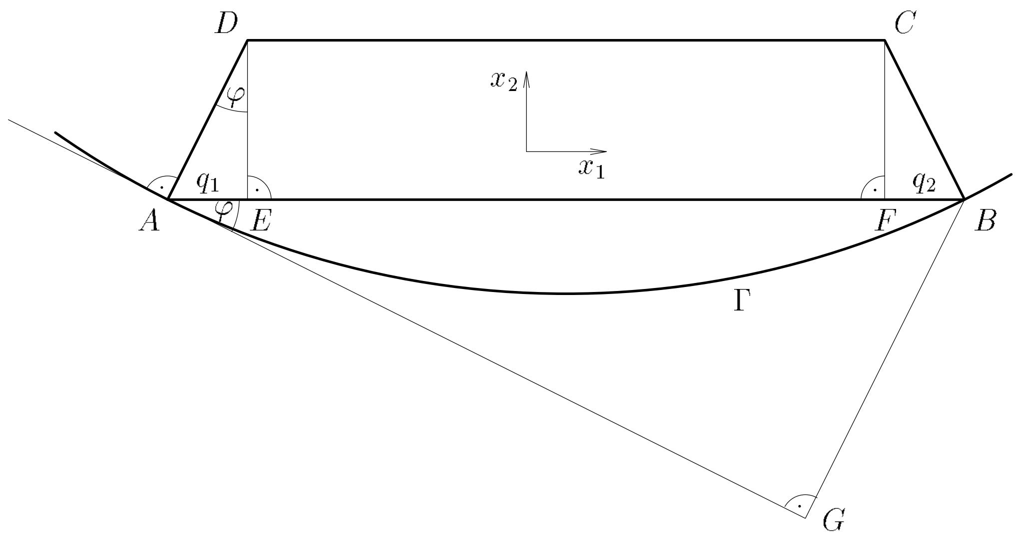

Figure 2: Anisotropic mesh in the boundary layer of a general polygonal domain.

Remark 2 We remark that the restriction (23) is very strong, but not without practical use. Consider an approximation of a curved C^2-boundary with anisotropic trapezoids e, see Figure 2 for an illustration. Describing e in a coordinate system where the long sides of e are parallel to the x_1-axis we see that $ah_2 = (\frac{1}{2}(q_1+q_2), 0)$ where $q_1 = \mathrm{meas}_1(\overline{AE})$ and $q_2 = \mathrm{meas}_1(\overline{FB})$. Viewing the boundary Γ in the tangential-normal coordinate system with respect to A we obtain $\mathrm{meas}_1(\overline{BG}) \le Ch_1^2$, $\mathrm{meas}_1(\overline{AG}) \sim h_1$, that means $\tan\varphi \le Ch_1$ and thus $q_1 = h_2 \tan\varphi \le Ch_1h_2$. The same can be derived for q_2. Therefore, a satisfies (23).

Corollary 5 *Of course one can set $h_2 \le h_1 =: h$ and derive*

$$\begin{aligned}
\|v - I_h^{(k)}v; L_p(e)\| &\le Ch^{k+1}|v; W^{k+1,p}(e)|, \\
|v - I_h^{(k)}v; W^{1,p}(e)| &\le C \begin{cases} \displaystyle\sum_{r=[k/2]+1}^{k+1} h^{r-1}|v; W^{r,p}(e)| = \mathcal{O}(h^{[k/2]}) & \textit{for } a \textit{ satisfying (12)}, \\ \displaystyle h^k \sum_{r=[k/2]+1}^{k+1} |v; W^{r,p}(e)| = \mathcal{O}(h^k) & \textit{for } a \textit{ satisfying (23)}, \end{cases}
\end{aligned}$$

and under higher regularity assumptions

$$|v - I_h^{(k)}v; W^{1,p}(e)| \le C \begin{cases} \displaystyle\sum_{r=[(k+1)/2]+1}^{k+2} h^{r-1}|v; W^{r,p}(e)| = \mathcal{O}(h^{[(k+1)/2]}) & \textit{for } a \textit{ from (12)}, \\ \displaystyle h^k|v; W^{k+1,p}(e)| + h^{k+1} \sum_{r=[(k+1)/2]+1}^{k+2} |v; W^{r,p}(e)| = \mathcal{O}(h^k) & \\ & \textit{for } a \textit{ from (23)}. \end{cases}$$

We note that $|v - I_h^{(1)}v; W^{1,2}(e)| \le Ch\,|v; W^{2,2}(e)|$ was derived in [25] under similar assumptions as in (12). This is a better result than in Corollary 5. It is based on a fully different proof.

4 Extension to three dimensions

In this section we will briefly comment on which parts of the theory developed in Sections 2 and 3 remain true for hexahedral elements. In analogy to the two dimensional case we restrict our considerations to elements $e = F(\hat{e})$ with trilinear mapping F. The notation is extended canonically.

The interpolation error estimates on $\hat{e}$ formulated in Theorem 1 are valid with the slight restriction that (2) holds only if $k \geq 2$ or $p > 2$, see [1].

All considerations of the affine transformation carry over, see [2]. In particular this concerns (6)–(9), and (10) with the restriction $p > 2$ for $k = 1$. For clarity, we formulate the definition of the mesh sizes and the conditions: Let E_e be the longest edge of e, and let F_e be the larger of the two faces of e with $E_e \subset \overline{F}_e$. Then we denote by $h_1 := \mathrm{meas}_1(E_e)$ the length of E_e, by $h_2 := \mathrm{meas}_2(F_e)/h_1$ the diameter of F_e perpendicularly to E_e, and by $h_3 := \mathrm{meas}_3(e)/(h_1 h_2)$ the diameter of e perpendicularly to F_e. For intermediate use we introduce another Cartesian coordinate system $(x_{1,e}, x_{2,e}, x_{3,e})$ such that $(0,0,0)$ is a vertex of $\hat{e}$, E_e is part of the $x_{1,e}$–axis, and F_e is part of the $x_{1,e}, x_{2,e}$–plane.

Interior angle condition (3D): There is a constant $\gamma_* < \pi$ (independent of h and $e \in \mathcal{T}_h$) such that the interior angles $\gamma_{i,j}$ in the faces as well as the angles γ_k between two faces of any element e are bounded by $\gamma_* : 0 < \gamma_* \leq \gamma_{i,j} \leq \pi - \gamma_*$, $i = 1, \ldots, 4$, $j = 1, \ldots, 6$, $0 < \gamma_* \leq \gamma_k \leq \pi - \gamma_*$, $k = 1, \ldots, 12$.

Coordinate system condition (3D): The transformation of the element coordinate system $(x_{1,e}, x_{2,e}, x_{3,e})$ into the system (x_1, x_2, x_3) can be determined as a translation and three rotations around the $x_{j,e}$-axes by angles ψ_j $(j = 1,2,3)$, where

$$|\sin\psi_1| \leq Ch_3/h_2, \quad |\sin\psi_2| \leq Ch_3/h_1, \quad |\sin\psi_3| \leq Ch_2/h_1.$$

In the isoparametric case we consider elements e which are a perturbation of brick elements. The conditions (11) and (12) read now

$$|a_i^{(j)}| \leq a_i h_3, \quad 0 \leq a_i \leq C, \quad i = 1,2,3, \; j = 1, \ldots, 8, \tag{25}$$

$$\frac{1}{2} - \frac{h_3}{h_1} a_1 - \frac{h_3}{h_2} a_2 - a_3 \geq a_0 > 0, \tag{26}$$

and Lemma 3 is valid for $i, j = 1,2,3$. The particular case (23) reads now

$$a_i \leq Ch_i, \quad i = 1,2,3. \tag{27}$$

While first and second order derivatives of F transform as in the two dimensional case we have now to consider also third order derivatives:

$$\left| \frac{\partial^2 x_i}{\partial \hat{x}_j \partial \hat{x}_k} \right| \leq 4 a_i h_3 (1 - \delta_{jk}), \quad i,j,k = 1,2,3, \tag{28}$$

$$\left| \frac{\partial^3 x_i}{\partial \hat{x}_1 \partial \hat{x}_2 \partial \hat{x}_3} \right| \leq 8 a_i h_3, \quad \frac{\partial^3 x_i}{\partial \hat{x}_j^2 \partial \hat{x}_k} = 0, \quad i,j,k = 1,2,3, \tag{29}$$

where δ_{ij} is the Kronecker delta. From this we get more terms in the transformation of $|\hat{D}^\alpha \hat{v}|$, $|\alpha| \geq 3$, which changes Theorem 4 to the following one.

Theorem 6 *Consider a brick element $\tilde{e}$ with sides of length h_1, h_2, and h_3, $h_1 \geq h_2 \geq h_3$, which are parallel to the axes of the x_1, x_2, x_3-coordinate system. The coordinates of the eight vertices are perturbed by vectors $a^{(i)} = (a_1^{(i)}, a_2^{(i)}, a_3^{(i)})^T$, $i = 1, \ldots, 8$, satisfying at least (25), (26). The resulting element is denoted by e. Then the following anisotropic interpolation error estimates hold:*

$$\|v - I_h^{(k)} v; L_p(e)\|^p \leq C \sum_{|\alpha|=k+1} h^{\alpha p} \|D^\alpha v; L_p(e)\|^p$$

for $v \in W^{k+1,p}(e)$, $1 \leq p \leq \infty$,

$$\begin{aligned} |v - I_h^{(k)} v; W^{1,p}(e)|^p &\leq C \sum_{|\alpha|=k} h^{\alpha p} |D^\alpha v; W^{1,p}(e)|^p + \\ &+ C \sum_{r=[k/3]+1}^{k} \sum_{\underline{i} \in E(k+1,r)} \sum_{|\alpha|=i_1} \sum_{|\beta|=i_2+i_3} h^{\alpha p} a^{\beta p} h_3^{(i_2+i_3-1)p} \|D^{\alpha+\beta} v; L_p(e)\|^p \end{aligned}$$

for $v \in W^{k+1,p}(e)$, $2 < p \leq \infty$, and

$$\begin{aligned} |v - I_h^{(k)} v; W^{1,p}(e)|^p &\leq C \sum_{k \leq |\alpha| \leq k+1} h^{\alpha p} |D^\alpha v; W^{1,p}(e)|^p + \\ &+ C \sum_{r=[(k+1)/3]+1}^{k+1} \sum_{\underline{i} \in E(k+2,r)} \sum_{|\alpha|=i_1} \sum_{|\beta|=i_2+i_3} h^{\alpha p} a^{\beta p} h_3^{(i_2+i_3-1)p} \|D^{\alpha+\beta} v; L_p(e)\|^p \end{aligned}$$

for $v \in W^{k+2,p}(e)$, $1 \leq p \leq \infty$.

The theorem can be proved with the same ideas as in the two-dimensional case. The only difference is that the set $E(m, r)$ can not be described in such an explicit form as in (22).

For the better understanding we formulate now the particular results for $k = 1$ and $k = 2$. We get for $v \in W^{k+1,p}(e)$

$$\begin{aligned} |v - I_h^{(1)} v; W^{1,p}(e)|^p &\leq C \sum_{|\alpha|=1} h^{\alpha p} \|D^\alpha v; W^{1,p}(e)\|^p \quad \text{for } p > 2 \text{ and } a \text{ satisfying (27)}, \\ |v - I_h^{(2)} v; W^{1,p}(e)|^p &\leq C \sum_{|\alpha|=2} h^{\alpha p} \|D^\alpha v; W^{1,p}(e)\|^p + C \sum_{|\alpha|=1} h^{\alpha p} \|D^\alpha v; L_p(e)\|^p \\ &\qquad \text{for } p \geq 1 \text{ and } a \text{ satisfying (27)}. \end{aligned}$$

If even $v \in W^{k+2,p}(e)$, $1 \leq p \leq \infty$, then

$$\begin{aligned} |v - I_h^{(1)} v; W^{1,p}(e)|^p &\leq C \sum_{1 \leq |\alpha| \leq 2} h^{\alpha p} |D^\alpha v; W^{1,p}(e)|^p + C \sum_{|\alpha|=1} h^{\alpha p} \|D^\alpha v; L_p(e)\|^p \\ &\qquad \text{for } a \text{ satisfying (27)}, \\ |v - I_h^{(2)} v; W^{1,p}(e)|^p &\leq C \begin{cases} \sum_{1 \leq |\alpha| \leq 3} h^{\alpha p} |D^\alpha v; W^{1,p}(e)|^p & \text{for } a \text{ satisfying (26)}, \\ \sum_{2 \leq |\alpha| \leq 3} h^{\alpha p} |D^\alpha v; W^{1,p}(e)|^p + \sum_{|\alpha|=2} h^{\alpha p} \|D^\alpha v; L_p(e)\|^p & \\ \qquad \text{for } a \text{ satisfying (27)}. \end{cases} \end{aligned}$$

For the general assumption (26) the three cases i) $k = 1$, $v \in W^{2,p}(e)$, ii) $k = 1$, $v \in W^{3,p}(e)$, and iii) $k = 2$, $v \in W^{3,p}(e)$ are not mentioned because we get no convergence. Note further that for $k = 2$, $v \in W^{k+2,p}(e)$ we get only first order convergence, if we do not restrict on (27). As to the author, a better result is possible only if special cases of the perturbation are considered. Moreover, observe that for $k = 1$ the use of (3) instead of (2) leads to more terms at the right hand side and needs higher regularity of v, but the estimate holds for all $p \geq 1$ and not only for $p > 2$.

Corollary 7 *Again, we can set $h_3 \le h_2 \le h_1 =: h$ and derive for a satisfying (26)*

$$\begin{aligned} \|v - I_h^{(k)} v; L_p(e)\| &\le C h^{k+1} |v; W^{k+1,p}(e)|, \\ |v - I_h^{(k)} v; W^{1,p}(e)| &\le C \sum_{r=[k/3]+1}^{k+1} h^{r-1} |v; W^{r,p}(e)| = \mathcal{O}(h^{[k/3]}), \end{aligned}$$

and under higher regularity assumptions

$$|v - I_h^{(k)} v; W^{1,p}(e)| \le C \sum_{r=[(k+1)/3]+1}^{k+2} h^{r-1} |v; W^{r,p}(e)| = \mathcal{O}(h^{[(k+1)/3]}).$$

For a satisfying (27) we obtain a better result but in a more complicated form:

$$\begin{aligned} |v - I_h^{(k)} v; W^{1,p}(e)| &\le C h^k |v; W^{k+1,p}(e)| + \\ &\quad + C \sum_{r=[k/3]+1}^{k} \sum_{\underline{i} \in E(k+1,r)} h^{k-i_3} |v; W^{r,p}(e)| = \mathcal{O}(h^{[(2k+1)/3]}), \end{aligned}$$

and for $v \in W^{k+2,p}(e)$

$$\begin{aligned} |v - I_h^{(k)} v; W^{1,p}(e)| &\le C \sum_{r=k}^{k+1} h^r |v; W^{r+1,p}(e)| + \\ &\quad + C \sum_{r=[(k+1)/3]+1}^{k+1} \sum_{\underline{i} \in E(k+2,r)} h^{k+1-i_3} |v; W^{r,p}(e)| = \mathcal{O}(h^{[(2k+3)/3]}). \end{aligned}$$

5 Anisotropic mesh refinement in boundary layers

Consider the reaction diffusion problem

$$-\varepsilon^2 \Delta u + cu = f \text{ in } \Omega \subset \mathbb{R}^2, \quad u = 0 \text{ on } \partial\Omega, \tag{30}$$

where Ω is a bounded polygonal domain, $\varepsilon \in (0,1]$ is the diffusion parameter, and c and f are sufficiently smooth functions, $c \ge c_0 > 0$. In the singularly perturbed case $\varepsilon \ll 1$ the solution of (30) is characterized by a boundary layer of width $\mathcal{O}(\varepsilon \ln \frac{1}{\varepsilon})$. For the analysis of the finite element method we need localized Sobolev norm estimates of the solution with respect to ε. Unfortunately, such estimates are hard to obtain. The results of Shishkin [21] for smooth domains and for the unit square lead us to an assumption which we are going to describe next.

Introduce a non-overlapping domain decomposition of Ω as illustrated in Figure 3. The subdomains are obtained by introducing lines with a distance $b := b_0 \varepsilon \ln \frac{1}{\varepsilon}$, $b_0 > \frac{3}{2c_0}$, to the boundary. The interior subdomain is denoted by Ω_1, the union of the small subdomains near the corners by $\Omega_2 = \bigcup_{\ell=1}^L \Omega_{2,\ell}$ and the union of all boundary strips by $\Omega_3 = \bigcup_{\ell=1}^L \Omega_{3,\ell}$. In Ω_3 we introduce a boundary fitted Cartesian coordinate system (x_1, x_2) with $x_2 := \operatorname{dist}(x, \partial\Omega)$; derivatives D^α are to be understood with respect to this coordinate system.

We assume that the following estimates hold:

$$\begin{aligned} |u; W^{2,2}(\Omega_1)|^2 &\le C && (31) \\ |u; W^{2,2}(\Omega_2)|^2 &\le C b \varepsilon^{-3} && (32) \\ \|D^\alpha u; L_2(\Omega_3)\|^2 &\le C(b\varepsilon^{2(2-|\alpha|)} + \varepsilon^{1-2\alpha_2}), \quad |\alpha| \le 3. && (33) \end{aligned}$$

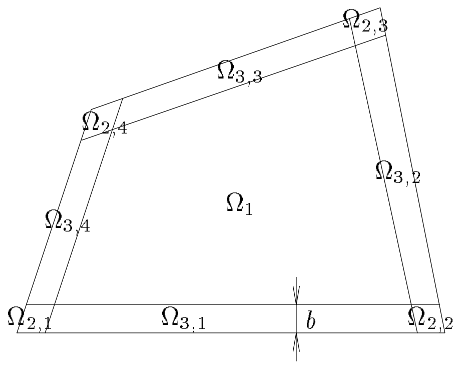

Figure 3: Illustration of the domain decomposition.

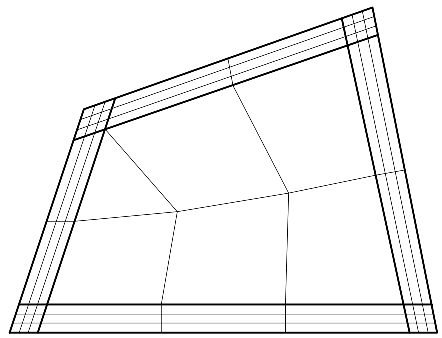

Figure 4: Anisotropic trapezoidal mesh in the boundary layer.

A discussion of these assumptions can be found in [3, Subsection 2.2].

With $V := W_0^{1,2}(\Omega)$ the variational formulation of problem (30) reads:

$$\textit{Find } u \in V \textit{ such that } a(u,v) = (f,v) \textit{ for all } v \in V, \tag{34}$$

where $a(u,v) := \varepsilon^2(\nabla u, \nabla v) + (cu, v)$ and $(.,.)$ is the $L_2(\Omega)$ inner product. Define by $|||\, v \,|||_\Omega := \sqrt{a(v,v)}$ the energy norm of $v \in V$.

For applying the finite element method each of the subdomains $\Omega_{2,\ell}$, $\Omega_{3,\ell}$, $\ell = 1,\ldots,L$, is subdivided into $\mathcal{O}(h^{-1}) \times \mathcal{O}(h^{-1})$ trapezoids, see Figure 4. The inner domain is classically meshed using isotropic triangles or quadrilaterals with mesh size h. Note that the anisotropic trapezoids in $\Omega_{3,\ell}$ satisfy relation (12) because $a_2 = 0$. We introduce now the finite element space $V_h \subset V \cap C(\overline{\Omega})$ of all continuous functions which are linear/bilinear in the triangular/quadrilateral elements e, respectively. Then the finite element solution of (30) is defined by:

$$\textit{Find } u_h \in V_h \textit{ such that } a(u_h, v_h) = (f, v_h) \textit{ for all } v_h \in V_h. \tag{35}$$

Theorem 8 *The finite element error of the problem described above can be estimated by*

$$|||\, u - u_h \,|||_\Omega \le Ch\left(\varepsilon^{1/2} \ln \tfrac{1}{\varepsilon} + h\right). \tag{36}$$

Proof In Ω_1 and Ω_2, isotropic elements with mesh size h and bh are used, respectively. From the standard theory we obtain with (31), (32)

$$\begin{aligned}
|||\, u - I_h^{(1)} u \,|||_{\Omega_1}^2 &\le C\|u - I_h^{(1)}u; L_2(\Omega_1)\|^2 + \varepsilon^2 |u - I_h^{(1)}u; W^{1,2}(\Omega_1)|^2 \\
&\le C(h^4 + \varepsilon^2 h^2)|u; W^{2,2}(\Omega_1)|^2 \;\le\; Ch^2(h^2 + \varepsilon^2), \\
|||\, u - I_h^{(1)} u \,|||_{\Omega_2}^2 &\le C\left((bh)^4 + \varepsilon^2 (bh)^2\right)|u; W^{2,2}(\Omega_2)|^2 \\
&\le C\varepsilon^4 h^2 \left(h^2 (\ln \tfrac{1}{\varepsilon})^4 + (\ln \tfrac{1}{\varepsilon})^2\right)\varepsilon^{-2} \ln \tfrac{1}{\varepsilon}.
\end{aligned}$$

In Ω_3 we have $h_1 \sim h$ and $h_2 \sim bh$. Using Theorem 4 and (33) we conclude for $\ell = 1,\ldots,L$

$$\begin{aligned}
|||\, u - I_h^{(1)} u \,|||_{\Omega_{3,\ell}}^2 &\le C \sum_{|\alpha|=2} h^{2\alpha} \|D^\alpha v; L_2(\Omega_{3,\ell})\|^2 + C\varepsilon^2 \sum_{1 \le |\alpha| \le 2} h^{2\alpha} |D^\alpha v; W^{1,2}(\Omega_{3,\ell})|^2 \\
&\le C\left(h^4 b + h^2 (bh)^2 \varepsilon^{-1} + (bh)^4 \varepsilon^{-3}\right) + \\
&\qquad + C\varepsilon^2 \left(h^2 \varepsilon^{-1} + (bh)^2 \varepsilon^{-3} + h^4 \varepsilon^{-1} + h^2 (bh)^2 \varepsilon^{-3} + (bh)^4 \varepsilon^{-5}\right) \\
&\le C\left(h^4 \varepsilon (\ln \tfrac{1}{\varepsilon})^4 + h^2 \varepsilon (\ln \tfrac{1}{\varepsilon})^2\right).
\end{aligned}$$

Summing up these estimates and using $|||\, u-u_h \,|||_\Omega = \inf_{v_h\in V_h} |||\, u-v_h \,|||_\Omega$ we get the assertion. $\square$

Note that the same result was obtained for triangular meshes in [3].

Remark 3 For comparison we point out the following: If the domain was meshed using isotropic elements of equal size h, then the error estimate would be $|||\, u-u_h \,|||_\Omega \leq Ch\varepsilon^{-1/2}$. For the proof one has to use a quasi-interpolant for $W^{1,2}(\Omega)$ functions.

If the boundary layer was resolved using isotropic elements of diameter bh then the estimate (36) can be proved with the same ideas as above. But then the number of elements grows to $\mathcal{O}(b^{-1}h^{-2}) = \mathcal{O}(\varepsilon^{-1}(\ln\frac{1}{\varepsilon})^{-1}h^{-2})$. That is an overrefinement and leads to a large increase of computational work.

As a third variant, assume the anisotropic mesh was applied as proposed above, but the anisotropic interpolation error estimates of Section 3 were not available. Then the error analysis of this section could use at best the estimate of [25], see also the comment after Corollary 5. This would lead to $|||\, u-u_h \,|||_{\Omega_3} \leq C(h^2+\varepsilon h)|u;W^{2,2}(\Omega_3)| \leq Ch(h+\varepsilon)\varepsilon^{-3/2}$, thus $|||\, u-u_h \,|||_\Omega \leq Ch(h+\varepsilon)\varepsilon^{-3/2}$.

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