Technische Universität Chemnitz-Zwickau

Sonderforschungsbereich 393

Numerische Simulation auf massiv parallelen Rechnern

Thomas Apel

Interpolation of non-smooth functions on anisotropic finite element meshes

Preprint SFB393/97-06

Abstract. In this paper, several modifications of the quasi-interpolation operator of Scott and Zhang (Math. Comp. 54(1990)190, 483–493) are discussed. The modified operators are defined for non-smooth functions and are suited for the application on anisotropic meshes. The anisotropy of the elements is reflected in the local stability and approximation error estimates. As an application, an example is considered where anisotropic finite element meshes are appropriate, namely the Poisson problem in domains with edges.

AMS(MOS) subject classification. 65D05, 65N30, 65N50

Key Words. Anisotropic finite elements, interpolation error estimate, quasi-interpolation, non-smooth functions, edge singularity, reaction diffusion problem.

Preprint-Reihe des Chemnitzer SFB 393

SFB393/97-06 **March 1997**

Contents

Author's address:

Thomas Apel
TU Chemnitz-Zwickau
Fakultät für Mathematik
D-09107 Chemnitz, Germany

apel@mathematik.tu-chemnitz.de
http://www.tu-chemnitz.de/~tap/

1 Introduction

Anisotropy The solution of elliptic boundary value problems may have *anisotropic behaviour* near certain manifolds $M \subset \overline{\Omega}$. That means that the solution varies significantly only perpendicularly to M. Examples include the Poisson problem in domains with concave edges M and singularly perturbed convection diffusion reaction problems where M is part of the boundary or an internal manifold. In such cases it is an obvious idea to reflect this anisotropy in the discretization by using *anisotropic meshes* with a small mesh size in the direction of the rapid variation of the solution and a larger mesh size in the perpendicular direction.

Consider an elliptic boundary value problem posed over a polyhedral domain $\Omega \subset I\!R^d$, $d = 2, 3$. We study the discretization error of the finite element method on a family of meshes $\mathcal{T}_h = \{e\}$ with the usual admissibility conditions (see, for example, Conditions $(\mathcal{T}_h1)$–$(\mathcal{T}_h5)$ in [15, Chapter 2]). Denote by h_e the diameter of the finite element e, and by ϱ_e the diameter of the largest inner ball of e. Then it is assumed in the classical finite element theory that $h_e \lesssim \varrho_e$, for the definition of $\lesssim$ see the end of this Introduction. This assumption is no longer valid in the case of anisotropic meshes. Conversely, anisotropic elements e are characterized by

$$\lim \frac{h_e}{\varrho_e} \to \infty$$

where the limit can be considered as of $h \to 0$ (see the application to the Poisson equation in [2, 8] or Section 6) or $\varepsilon \to 0$ where ε is some (small perturbation) parameter of the problem (see the singularly perturbed problems in [4, 5]).

Interpolation Let $V_h := \{v_h \in W^{1,2}(\Omega) : v_h|_e \in \mathcal{P}_{k,e} \text{ for all } e \in \mathcal{T}_h\}$ be the finite element space, a space of piecewise polynomial functions ($\mathcal{P}_{k,e}$ is introduced at Page 4) on the family of meshes under consideration. Then the estimation of the finite element error is reduced by Ceá's lemma to a general approximation problem of the exact solution u in V_h. For Lagrangian finite elements, the simplest approximate is the *nodal interpolant*

$$\mathrm{I}_h u := \sum_{i \in I} u(X_i)\varphi_i(x) \tag{1.1}$$

where X_i are the nodes and $\varphi_i(x)$ are the *nodal basis functions*:

$$\varphi_i(X_j) = \delta_{ij}, \quad i, j \in I. \tag{1.2}$$

Because I_h is defined locally on every element the interpolation error $u - \mathrm{I}_h u$ can be estimated elementwise. Before we discuss the drawback of the nodal interpolant we shall recall the *anisotropic interpolation error estimates.* We denote error estimates as anisotropic if they are sharp enough to reflect the different mesh sizes and not only the largest diameter.

For simplicity in this Introduction consider a triangle or a tetrahedron $e \subset I\!R^d$ with mesh sizes $h_1, \ldots, h_d$ as given in Figure 1. That means that the element e has d edges of length h_i which are parallel to the corresponding coordinate axes. Then for linear elements the following estimates hold [2, 24]:

$$\|u - \mathrm{I}_h u; L^p(e)\| \lesssim \sum_{|\alpha|=2} h^\alpha \|D^\alpha u; L^p(e)\|, \quad p \in [1, \infty], \tag{1.3}$$

$$|u - \mathrm{I}_h u; W^{1,p}(e)| \lesssim \sum_{|\alpha|=1} h^\alpha |D^\alpha u; W^{1,p}(e)|, \quad d = 2 \text{ or } p \in (2, \infty]. \tag{1.4}$$

For the notation see the end of this Introduction.

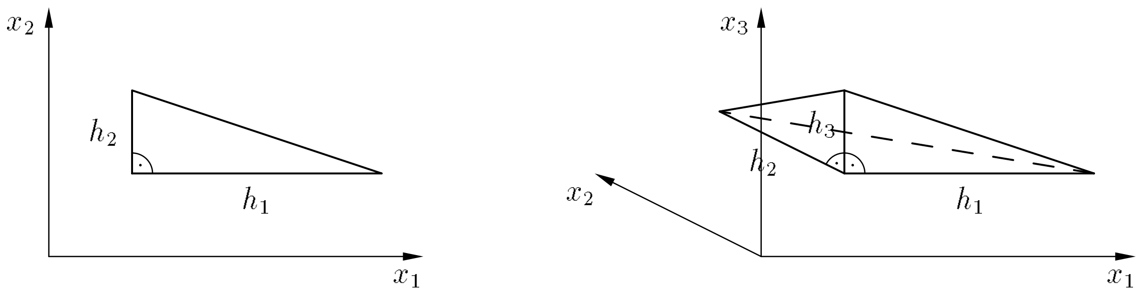


Figure 1: Illustration of the simplest anisotropic finite elements

In the sequel, we will call an *estimate* to be *of type* (m, n) if certain m-th derivatives (left hand side) are estimated against n-th derivatives of the solution. In this sense Estimate (1.3) is of type $(0, 2)$.

Quasi-interpolation The aim of this paper is to investigate (several slightly different) more general approximates $\mathrm{Q}_h u \in V_h$ (sometimes called *quasi-interpolants*) which shall not have some of the disadvantages of $\mathrm{I}_h u$. This includes:

1. $\mathrm{Q}_h u$ shall be defined (at least) for $u \in W^{1,2}(\Omega)$ where pointwise values may not be well-defined.
2. The restriction $p > 2$ for $d = 3$, see (1.4), shall not be necessary in the approximation error estimate of type $(1, 2)$.
3. $\mathrm{Q}_h u$ shall allow estimates of type $(0, 1)$ and, if possible, of type $(1, 1)$.

Of course, some favourable properties of $\mathrm{I}_h u$ should be preserved:

4. $\mathrm{Q}_h u$ shall be defined locally. This means, that $(\mathrm{Q}_h u)(x)$ with $x \in e$ shall depend only on the values of u in a small neighbourhood S_e of e, where S_e consists of a finite number (independent of h) of elements of $\mathcal{T}_h$. (For the interpolant we had in particular $S_e = e$.)
5. Q_h shall reproduce piecewise polynomials: $\mathrm{Q}_h u_h = u_h$ for all $u_h \in V_h$.

For *isotropic meshes* such operators have been studied in the literature. For an introduction, denote by $\varphi_i \in V_h$ the nodal basis functions in V_h and define

$$\mathrm{Q}_h u := \sum_{i \in I} a_i \varphi_i \tag{1.5}$$

with real numbers a_i still to be specified. Note that $\mathrm{Q}_h = \mathrm{I}_h$ if $a_i = u(X_i)$ for all $i \in I$.

In order to treat non-smooth functions the idea is to consider subdomains $\sigma_i \subset \overline{\Omega}$ (their choice will be discussed later), to define an L^2-projection operator

$$\Pi_{\sigma_i} : L^2(\sigma_i) \to \mathcal{P}_{k,\sigma_i}, \tag{1.6}$$

and to choose

$$a_i := (\Pi_{\sigma_i} u)(X_i), \tag{1.7}$$

for more details see (2.1)–(2.3). The numbers a_i can be considered as averaged values of u in X_i. Different authors chose different σ_i resulting in different quasi-interpolation operators. We will now introduce three of them. For unambiguous reference we distinguish them by different symbols, C_h, O_h, and Z_h.

Clement [16] uses $\overline{\sigma_i} := \bigcup_{\overline{e}\ni X_i} \overline{e}$. The resulting operator C_h,

$$(\mathrm{C}_h u)(x) := \sum_{i\in I} (\Pi_{\sigma_i} u)(X_i) \cdot \varphi_i(x),$$

is even defined for $u \in L^1(\Omega)$ and allows estimates of type (m,ℓ) for all $0 \le m \le \ell \le k+1$, $k \ge 1$ is defined below. However, the operator C_h in this original form does not satisfy Property 5, but this can be corrected by defining

$$\Pi_{\sigma_i} : L^2(\sigma_i) \to V_h|_{\sigma_i}. \tag{1.8}$$

A modification of the Clement operator is discussed by Oswald [23]. For defining σ_i, he fixes just one (arbitrary) element $e =: \sigma_i$ with $X_i \in \overline{e}$. The resulting operator O_h allows the same estimates as C_h, but we have $V_h|_{\sigma_i} = \mathcal{P}_{k,\sigma_i}$. Some more details on C_h and O_h are given at the end of Section 2 when more notation has been introduced and more ideas have been developed.

The disadvantage of both C_h and O_h is that $\mathrm{C}_h u$ and $\mathrm{O}_h u$ do not satisfy the same Dirichlet boundary conditions as u does. For this reason, Scott and Zhang [25] modified again the choice of σ_i and used not only d-dimensional subdomains σ_i but also $(d-1)$-dimensional ones. In particular, they chose $\sigma_i \subset \partial\Omega$ if $X_i \in \partial\Omega$. Because we exploit this idea in this paper we will introduce the resulting operator Z_h in more detail in Section 2. Using lower-dimensional subdomains σ_i we are able to define in Sections 3–5 further operators S_h, L_h, and E_h and to prove estimates of type (m,ℓ) for *anisotropic meshes.* Some of the results were derived independently by Becker [12].

Elements of tensor product type Let $\hat{e}$ be a reference element. In the cases of triangles ($\hat{e} := \{(\hat{x}_1,\hat{x}_2) \in I\!R^2 : 0 < \hat{x}_1 < 1, 0 < \hat{x}_2 < 1-\hat{x}_1\}$), rectangles ($\hat{e} := \{(\hat{x}_1,\hat{x}_2) \in I\!R^2 : 0 < \hat{x}_1,\hat{x}_2 < 1\}$), pentahedra ($\hat{e} := \{(\hat{x}_1,\hat{x}_2,\hat{x}_3) \in I\!R^3 : 0 < \hat{x}_1,\hat{x}_3 < 1, 0 < \hat{x}_2 < 1-\hat{x}_1\}$), and hexahedra ($\hat{e} := \{(\hat{x}_1,\hat{x}_2,\hat{x}_3) \in I\!R^3 : 0 < \hat{x}_1,\hat{x}_2,\hat{x}_3 < 1\}$) it is sufficient to consider one unique $\hat{e}$. Only for tetrahedra we consider two reference elements: $\hat{e} := \{(\hat{x}_1,\hat{x}_2,\hat{x}_3) \in I\!R^3 : 0 < \hat{x}_1 < 1, 0 < \hat{x}_2 < 1-\hat{x}_1, 0 < \hat{x}_3 < 1-\hat{x}_1-\hat{x}_2\}$ for elements with a face parallel to the x_1,x_2-plane and $\hat{e} := \{(\hat{x}_1,\hat{x}_2,\hat{x}_3) \in I\!R^3 : 0 < \hat{x}_1 < 1, 0 < \hat{x}_2 < 1-\hat{x}_1, \hat{x}_1 < \hat{x}_3 < 1-\hat{x}_2\}$ for elements without such a face.

In this paper, we treat *affine finite elements of tensor product type*, that means, the transformation of a reference element $\hat{e}$ to the element e shall have (block) diagonal form,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \pm h_{1,e} & 0 \\ 0 & \pm h_{2,e} \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} + b_e \quad \text{for } d = 2, \tag{1.9}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} B_e & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & \pm h_{d,e} \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix} + b_e \quad \text{for } d = 3, \tag{1.10}$$

where $b_e \in I\!R^2$ and $B_e \in I\!R^{2\times 2}$ with

$$|\det B_e| \sim h_{1,e}^2, \quad \|B_e\| \sim h_{1,e}, \quad \|B_e^{-1}\| \sim h_{1,e}^{-1}. \tag{1.11}$$

In this way the element sizes $h_{1,e}, \ldots, h_{d,e}$ are implicitly defined. Note that (1.11) yields $h_{1,e} \sim h_{2,e}$ for three-dimensional elements. Up to now we did not assume a relation between $h_{1,e}$ and $h_{d,e}$. But in Section 3 we will consider the case $h_{1,e} \lesssim h_{d,e}$ (interesting is $h_{1,e} = o(h_{d,e})$) and in Sections 4 and 5 we will examine $h_{d,e} \lesssim h_{1,e}$. Note further that under these assumptions the triangles/tetrahedra can be grouped into pairs/triples which form a rectangle/pentahedron of tensor product type. We will use this property in Section 3.

We demand that there is no abrupt change in the element sizes, that means, the relation

$$h_{i,e} \sim h_{i,e'} \quad \text{for all } e' \text{ with } \quad \overline{e} \cap \overline{e'} \neq \emptyset \tag{1.12}$$

holds for $i = 1, \ldots, d$.

The set of shape functions $\mathcal{P}_{k,e}$,

$$\mathcal{P}_{k,e} \supset \mathcal{P}_k^d := \left\{ \sum_{|\alpha| \leq k} a_\alpha x^\alpha; \quad x = (x_1, \ldots, x_d) \right\}, \tag{1.13}$$

is defined as usual, that means, $\mathcal{P}_{k,e} = \mathcal{P}_k^d$ for the simplicial elements, $\mathcal{P}_{k,e} = (\mathcal{P}_k^1)^d$ for quadrilateral and hexahedral elements, and $\mathcal{P}_{k,e} = \mathcal{P}_k^2 \times \mathcal{P}_k^1$ for pentahedral elements. The multi-index notation used in (1.13) is explained at the end of this section. Moreover, for a simple notation later on we define $\mathcal{P}_{-1}^d := \{0\}$.

Outline In Section 2 we will recall the original Scott-Zhang operator Z_h, derive some anisotropic estimates of type $(0, \ell)$, $1 \leq \ell \leq k+1$, and show that the operator Z_h has to be modified for error estimates of type $(1, \ell)$. Sections 3–5 are devoted to the study of operators S_h, L_h, and E_h which are different modifications of Z_h. These operators allow stability and approximation estimates of type (m, ℓ) for different ranges of m and ℓ. There are also differences in the applicability of these operators concerning the types of elements and the ability to satisfy Dirichlet boundary conditions. We will summarize this in Section 7.

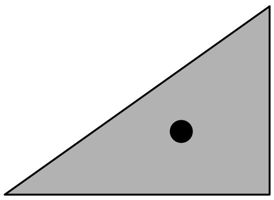
Before, in Section 6, we shall apply the operators S_h and E_h and derive finite element error estimates for the Poisson problem in certain domains with edges. The result can not be obtained using the nodal interpolation operator I_h. This underlines the importance of this study.

Some notation Let d be the space dimension, $x = (x_1, \ldots, x_d)$ the global Cartesian coordinate system, and $h_{1,e}, \ldots, h_{d,e}$ the element sizes, see (1.9)–(1.11). In view of (1.12) and because most considerations in this paper are local, we will often omit the second subscript. Moreover, we denote uniformly in the whole paper by

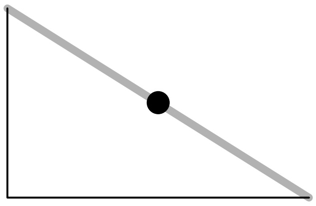
e	a finite element,
S_e	the patch of elements around e, see (2.6),
X_i	the nodes of the mesh, $i \in I$,
φ_i	the nodal shape functions, $\varphi_i(X_j) = \delta_{ij}$,
σ_i	a subdomain related to X_i (different for C_h, O_h, Z_h, S_h, L_h, and E_h),
k	the degree of the shape functions in the sense of (1.13),
Π_{σ_i}	the projection operator $L^2(\sigma_i) \to \mathcal{P}_{k,\sigma_i}$,
I_h	the nodal **i**nterpolation operator,
Q_h	a general **q**uasi-interpolation operator,
C_h	the **C**lement operator,
O_h	the quasi-interpolation operator introduced by **O**swald,
Z_h	the original Scott-**Z**hang operator,
S_h	the modified Scott-Zhang operator using **s**hort edges(2D)/faces(3D),
L_h	the modified Scott-Zhang operator using **l**ong edges(2D)/faces(3D),
E_h	the modified Scott-Zhang operator using long **e**dges (3D).

We use a multi-index notation with $\alpha := (\alpha_1, \ldots, \alpha_d)$, α_i non-negative integers,

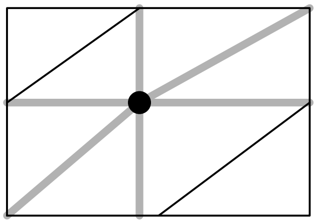
$$|\alpha| := \sum_{i=1}^d \alpha_i, \quad h^\alpha := h_1^{\alpha_1} \cdots h_d^{\alpha_d}, \text{ and } D^\alpha := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}.$$



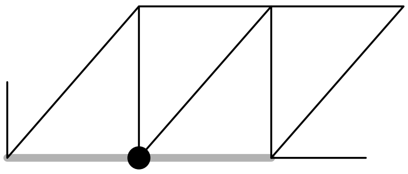
(a) X_i is an interior point of an element.



(b) X_i is an interior point of an edge.



(c) X_i is a vertex within the domain (here: 6 possibilities for σ_i).



(d) X_i is a vertex at the boundary (2 possibilities for σ_i).

Figure 2: Choice of σ_i in dependence on X_i for the definition of Z_h.

$W^{\ell,p}(e)$ ($\ell \in \mathbb{N}_0$, $p \in [1,\infty]$) are the Sobolev spaces with

$$\|v; W^{\ell,p}(e)\|^p := \sum_{|\alpha|\le\ell} \int_e |D^\alpha v|^p, \quad |v; W^{\ell,p}(e)|^p := \sum_{|\alpha|=\ell} \int_e |D^\alpha v|^p$$

for $p < \infty$ and the usual modification for $p = \infty$.

The notation $a \lesssim b$ and $a \sim b$ means the existence of positive constants C_1 and C_2 (which are independent of $\mathcal{T}_h$ and of the function under consideration) such that $a \le C_2 b$ and $C_1 b \le a \le C_2 b$, respectively.

2 The original Scott-Zhang operator Z_h

In this section we will recall the operator Z_h defined by Scott and Zhang [25] and examine to what extend anisotropic error estimates can be derived by simply carrying out the transformations more carefully. We will see that estimates of type $(0,\ell)$ are valid, but modifications of the operator are necessary for estimates of derivatives of the approximation error.

As introduced in Section 1 we define $\mathrm{Z}_h u$ via numbers $a_i = (\Pi_{\sigma_i} u)(X_i)$, where Π_{σ_i} is a projection operator with respect to a certain subdomain σ_i, $i \in I$. The subdomains σ_i are chosen by the following rules (see also Figure 2 for the case of triangles).

- If the node X_i is an *interior point* of an element $e_0 \subset \mathcal{T}_h$ then $\sigma_i := e_0$.
- Otherwise X_i is a *boundary point* of one or more elements $e \subset \mathcal{T}_h$, and σ_i is chosen as some $(d-1)$-dimensional edge/face ς of one of these elements:
  - If there is an edge/face ς so that X_i is an *interior point* of ς, then σ_i is uniquely determined by $\sigma_i := \varsigma$.
  - If not, then σ_i is taken as one of the edges/faces with $X_i \in \bar{\varsigma}$. However, we restrict this choice in the case $X_i \in \partial\Omega$ by demanding $\sigma_i \subset \partial\Omega$ then.

The $L^2(\sigma_i)$-projection $\Pi_{\sigma_i} u \in V_h|_{\sigma_i}$ is defined by

$$\|u - \Pi_{\sigma_i} u; L^2(\sigma_i)\| = \min_{v \in V_h|_{\sigma_i}} \|u - v; L^2(\sigma_i)\|. \tag{2.1}$$

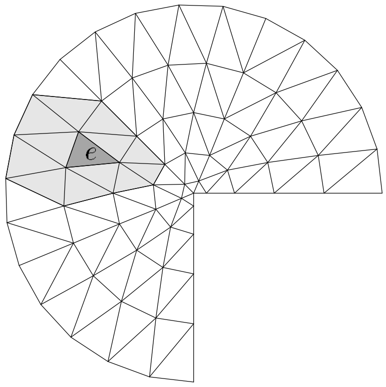


Figure 3: Illustration of S_e in a two-dimensional example.

An explicit representation of $(\Pi_{\sigma_i} u)(X_i)$ can be given by introducing the (unique) function $\psi_i \in V_h|_{\sigma_i}$ with

$$\int_{\sigma_i} \psi_i \varphi_j = \delta_{ij} \quad \text{for all } j \in I. \tag{2.2}$$

Then one finds easily that

$$(\Pi_{\sigma_i} u)(X_i) = \int_{\sigma_i} u\psi_i. \tag{2.3}$$

To see this recall that a projection operator $P : X \to Y \subset X$ can be defined via $Pu = \sum_j (u, \psi_j)_X \, \varphi_j$ where $\{\varphi_j\}$ is a basis in Y and $\{\psi_j\}$ is the corresponding biorthogonal basis with respect to the scalar product $(.,.)_X$ in X. As already mentioned in Section 1, see (1.5) and (1.7), the Scott-Zhang operator Z_h is now defined as

$$Z_h u := \sum_i (\Pi_{\sigma_i} u)(X_i) \cdot \varphi_i = \sum_i \left(\int_{\sigma_i} u\psi_i \right) \cdot \varphi_i. \tag{2.4}$$

Though Π_{σ_i} is defined by (2.1) for $u \in L^2(\sigma_i)$, this approach can be extended to functions $u \in L^1(\sigma_i)$ because the polynomial function ψ_i is from $L^\infty(\sigma_i)$ such that the integral in (2.3) is finite. That means that the approximation operator $Z_h : W^{\ell,p}(\Omega) \to V_h$ can be defined for

$$\ell \geq 1 \quad \text{for } p = 1, \qquad \ell > \frac{1}{p} \quad \text{otherwise.} \tag{2.5}$$

The restrictions to ℓ and p in (2.5) follow from a trace theorem and guarantee that $u|_{\sigma_i} \in L^1(\sigma_i)$ also for $(d-1)$-dimensional σ_i. In this paper, we consider only integer ℓ, therefore (2.5) is equivalent to

$$\ell \geq 1, \quad p \in [1, \infty].$$

Note further that the approximation operator Z_h does not only preserve homogeneous Dirichlet boundary conditions but also inhomogeneous conditions $u = g$ on $\partial\Omega$ (at least in the sense of $L^1(\partial\Omega)$) if $g \in V_h|_{\partial\Omega}$.

Denote by

$$S_e := \operatorname{int} \bigcup \{ \overline{e'} : e' \in \mathcal{T}_h, \overline{e'} \cap \overline{e} \neq \emptyset \} \tag{2.6}$$

the patch of elements around e and note that $\sigma_i \subset S_e$ for all i with $X_i \in \overline{e}$, see also the illustration in Figure 3. (The mesh in the figure is not of tensor product type but in [25] this was not required.) For isotropic *simplicial* elements e ($h_1 \sim \ldots \sim h_d$) Scott and Zhang

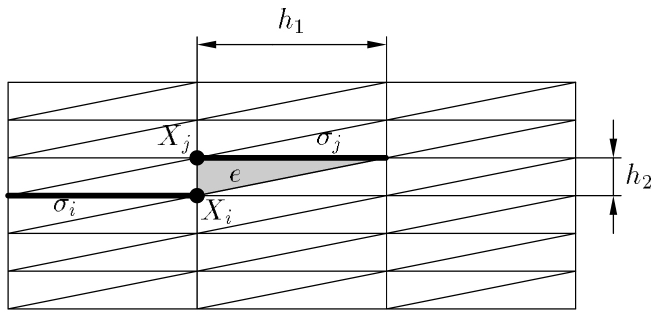


Figure 4: Illustration of the counterexample.

proved the following stability and approximation result [25]: If $1 \le \ell \le k+1$ and $p \in [1,\infty]$ then the estimates

$$|Z_h u; W^{m,p}(e)| \lesssim \sum_{j=0}^{\ell} h_1^{j-m} |u; W^{j,p}(S_e)| \tag{2.7}$$

$$|u - Z_h u; W^{m,p}(e)| \lesssim h_1^{\ell-m} |u; W^{\ell,p}(S_e)| \tag{2.8}$$

hold for $0 \le m \le \ell$. Recall that k corresponds to the degree of the polynomials, see (1.13). The anisotropic estimate corresponding to (2.8) would be

$$|u - Z_h u; W^{m,p}(e)| \lesssim \sum_{|\alpha|=\ell-m} h^\alpha |D^\alpha u; W^{m,p}(S_e)| \tag{2.9}$$

which obviously does *not* hold for $m \ge 1$ in the general setting of σ_i as introduced above, see Example 1. But we will prove in Theorem 3 that (2.9) holds for $m = 0$.

Example 1 In this example we will show that (2.9) does in general not hold in the case $m = k = 1$ and the whole range of ℓ, namely $\ell = 1, 2$. Consider the situation as illustrated in Figure 4, and let $u = u(x_1)$ be any function which is independent of the variable x_2. This leads to $a_i \ne a_j$, where a_i and a_j are independent of h_2, that means

$$\left.\frac{\partial Z_h u}{\partial x_2}\right|_e = h_2^{-1} f(u, x_1, h_1)$$

with a certain function f. In view of $\frac{\partial u}{\partial x_2} = 0$ we obtain

$$\begin{aligned} |u - Z_h u; W^{1,p}(e)| &\ge \left\| \frac{\partial Z_h u}{\partial x_2}, L^p(e) \right\| = h_2^{-1+1/p} F(u, x_1, h_1), \\ \sum_{|\alpha|=\ell-1} h^\alpha |D^\alpha u; W^{1,p}(S_e)| &= h_1^{\ell-1} \left\| \frac{\partial^\ell u}{\partial x_1^\ell}, L^p(S_e) \right\| = h_2^{1/p} G(u, x_1, h_1). \end{aligned}$$

Consequently, for $f(u, x_1, h_1) \ne 0$ (which is the case in general) and $h_2 = h_1^s$ with sufficiently large s (depending on u) Estimate (2.9) can not be satisfied.

Before we formulate Theorem 3 we will prove a lemma which is useful not only in the proof of Theorem 3 but also in the next sections. The lemma has similarities to the Bramble-Hilbert theory which was developed in [13, 14] for isotropic elements and extended in [2] to anisotropic elements. Here, the difference is that (in general) S_e can not be transformed by an affine mapping to a reference configuration $\hat{S}$. The isotropic version of Lemma 1 is proved in [25] using results from [17] and can easily be generalized to our case.

Lemma 1 *For any $u \in W^{\ell,p}(S_e)$ there exists a polynomial $w \in \mathcal{P}^d_{\ell-1}$ such that*

$$\sum_{|\alpha|\le \ell-m} h^\alpha |D^\alpha(u-w); W^{m,p}(S_e)| \lesssim \sum_{|\alpha|=\ell-m} h^\alpha |D^\alpha u; W^{m,p}(S_e)|,$$

for all $m = 0,\dots,\ell$.

Proof By the change of variables $x_i = \tilde{x}_i h_i$ we transform S_e to $\tilde{S}_e$. According to (1.12) and the tensor product character of our mesh we realize that $\tilde{S}_e$ has a diameter of order one. Moreover, $\tilde{S}_e$ is star-shaped with respect to a ball B_1 with $\operatorname{diam} B_1 \sim 1$, or $\tilde{S}_e$ is at least the union of a finite collection of (overlapping) domains $\tilde{S}_{e,j}$ that are star-shaped with respect to a balls B_j with $\operatorname{diam} B_j \sim 1$. Let $B \subset \tilde{S}_e$ be any ball with $\operatorname{diam} B \sim 1$, choose a function $\phi \in C_0^\infty(B)$ with integral one, and define

$$\tilde{w}(\tilde{x}) := \sum_{|\alpha|\le \ell-1} \int_B \phi(\tilde{y}) \cdot (\tilde{D}^\alpha \tilde{u})(\tilde{y}) \cdot \frac{(\tilde{x}-\tilde{y})^\alpha}{\alpha!}\, \mathrm{d}\tilde{\mathrm{y}} \ \in \mathcal{P}^{\mathrm{d}}_{\ell-1},$$

$\tilde{x} = (\tilde{x}_1,\dots,\tilde{x}_d)$, $\tilde{y} = (\tilde{y}_1,\dots,\tilde{y}_d)$, $\alpha! = \alpha_1!\cdots\alpha_d!$. We can now apply Theorem 4.2 of [17] with $\mathcal{A} = \{\alpha \in \mathbb{N}_0^d : |\alpha| \le \ell\}$, and obtain for all β with $|\beta| = m$, $0 \le m \le \ell-1$,

$$\|\tilde{D}^\beta(\tilde{u}-\tilde{w}); W^{\ell-m-1,p}(\tilde{S}_e)\| \lesssim |\tilde{D}^\beta \tilde{u}; W^{\ell-m,p}(\tilde{S}_e)|.$$

By transforming this estimate to S_e and summing up over all β we conclude

$$\begin{aligned}
\sum_{|\alpha|\le \ell-m-1} h^\alpha \|D^{\alpha+\beta}(u-w); L^p(S_e)\| &\lesssim \sum_{|\alpha|=\ell-m} h^\alpha \|D^{\alpha+\beta} u; L^p(S_e)\|,\\
\sum_{|\alpha|\le \ell-m-1} h^\alpha |D^\alpha(u-w); W^{m,p}(S_e)| &\lesssim \sum_{|\alpha|=\ell-m} h^\alpha |D^\alpha u; W^{m,p}(S_e)|.
\end{aligned}$$

Because of $D^\gamma w = 0$ for $|\gamma| = \ell$ the sum on the left hand side can be extended to $|\alpha| \le \ell - m$. □

Corollary 2 *Let $m_1 + m_2 = m \le \ell$. For any $u \in W^{\ell,p}(S_e)$ there exists a polynomial $w \in \mathcal{P}^d_{m-1}$ such that*

$$\sum_{|\alpha|\le m_2} \sum_{|\beta|\le \ell-m} h^{\alpha+\beta} |D^{\alpha+\beta}(u-w); W^{m_1,p}(S_e)| \lesssim \sum_{|\alpha|=m_2} \sum_{|\beta|\le \ell-m} h^{\alpha+\beta} |D^{\alpha+\beta} u; W^{m_1,p}(S_e)|.$$

Proof We reformulate the left hand side and split it in two terms.

$$\begin{aligned}
\sum_{|\alpha|\le m_2} \sum_{|\beta|\le \ell-m} h^{\alpha+\beta} |D^{\alpha+\beta}(u-w); W^{m_1,p}(S_e)| &\sim \sum_{|\delta|\le \ell-m_1} h^\delta |D^\delta(u-w); W^{m_1,p}(S_e)|\\
= \sum_{|\delta|\le m_2} h^\delta |D^\delta(u-w); W^{m_1,p}(S_e)| &+ \sum_{m_2<|\delta|\le \ell-m_1} h^\delta |D^\delta(u-w); W^{m_1,p}(S_e)|
\end{aligned}$$

In view of $m_2 = m - m_1$, the first term can be estimated via Lemma 1. The second term contains only derivatives of order higher than m, that means that w plays no role. Consequently, w can be chosen such that

$$\begin{aligned}
&\sum_{|\alpha|\le m_2} \sum_{|\beta|\le \ell-m} h^{\alpha+\beta} |D^{\alpha+\beta}(u-w); W^{m_1,p}(S_e)|\\
&\lesssim \sum_{|\delta|=m_2} h^\delta |D^\delta u; W^{m_1,p}(S_e)| + \sum_{m_2<|\delta|\le \ell-m_1} h^\delta |D^\delta u; W^{m_1,p}(S_e)|\\
&\lesssim \sum_{|\alpha|=m_2} h^\alpha |D^\alpha u; W^{m_1,p}(S_e)| + \sum_{|\alpha|=m_2} \sum_{1\le|\beta|\le \ell-m} h^{\alpha+\beta} |D^{\alpha+\beta} u; W^{m_1,p}(S_e)|,
\end{aligned}$$

and the corollary is proved. □

Theorem 3 *On anisotropic meshes of tensor product type the Scott-Zhang approximation operator* Z_h *satisfies the following stability and approximation error estimates of type* $(0,\ell)$*:*

$$\|\mathrm{Z}_h u; L^q(e)\| \lesssim (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha|\le\ell} h^\alpha \|D^\alpha u; L^p(S_e)\|, \tag{2.10}$$

$$\|u - \mathrm{Z}_h u; L^q(e)\| \lesssim (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha|=\ell} h^\alpha \|D^\alpha u; L^p(S_e)\|, \tag{2.11}$$

$\ell = 1,\dots,k+1$, *provided that* $u \in W^{\ell,p}(S_e)$*. For* (2.11) *the numbers* $p,q \in [1,\infty]$ *and* $\ell \in I\!N$ *must be such that* $W^{\ell,p}(e) \hookrightarrow L^q(e)$*.*

Proof We start by concluding from $\int_{\sigma_i} \varphi_i\psi_i = 1$ and $\|\varphi_i; L^\infty(\sigma_i)\| = 1$ that

$$\|\psi_i; L^\infty(\sigma_i)\| \sim (\operatorname{meas}\sigma_i)^{-1}. \tag{2.12}$$

Using the definition of $\mathrm{Z}_h u$ we find with (2.12) that

$$\begin{aligned}
\|\mathrm{Z}_h u; L^q(e)\| &\le \sum_{i\in I_e} \left\| \varphi_i \int_{\sigma_i} u\psi_i; L^q(e) \right\| \\
&\le (\operatorname{meas} e)^{1/q} \sum_{i\in I_e} \left| \int_{\sigma_i} u\psi_i \right| \\
&\lesssim (\operatorname{meas} e)^{1/q} \sum_{i\in I_e} (\operatorname{meas}\sigma_i)^{-1} \|u; L^1(\sigma_i)\|,
\end{aligned}$$

where I_e is the set of nodes contained in $\overline{e}$. If σ_i has the same dimension as e (that means X_i is an inner node of e and $\sigma_i = e$) then we use the Hölder inequality and find

$$\begin{aligned}
\|u; L^1(\sigma_i)\| &\le (\operatorname{meas} e)^{1-1/p} \|u; L^p(\sigma_i)\| \\
&\lesssim \operatorname{meas}\sigma_i\, (\operatorname{meas} e)^{-1/p} \|u; L^p(S_e)\|.
\end{aligned} \tag{2.13}$$

If σ_i has lower dimension we use the trace theorem $W^{\ell,p}(S_e) \hookrightarrow W^{\ell,p}(e') \hookrightarrow L^1(\sigma_i)$ ($e' \subset S_e$ is an element with $\sigma_i \subset \overline{e'}$) in the form

$$\|u; L^1(\sigma_i)\| \lesssim \operatorname{meas}\sigma_i (\operatorname{meas} e)^{-1/p} \sum_{|\alpha|\le\ell} h^\alpha \|D^\alpha u; L^p(S_e)\| \tag{2.14}$$

which holds for $\ell \ge 1$. Combining the last three estimates we obtain the stability estimate (2.10). From this we derive for any $w \in \mathcal{P}^d_{\ell-1} \subset \mathcal{P}^d_k$

$$\begin{aligned}
\|u - \mathrm{Z}_h u; L^q(e)\| &\le \|u - w; L^q(e)\| + \|\mathrm{Z}_h(u-w); L^q(e)\| \\
&\lesssim (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha|\le\ell} h^\alpha \|D^\alpha(u-w); L^p(S_e)\|
\end{aligned}$$

where we used the embedding $W^{\ell,p}(e) \hookrightarrow L^q(e)$. With Lemma 1 we conclude (2.11). $\square$

In the remaining part of this section we will discuss to what extend the previous results carry over to the operators C_h and O_h which were considered by Clement [16] and Oswald [23] for isotropic meshes. Recall from the Introduction that the difference between Z_h, C_h, and O_h is only in the definition of the subdomains σ_i. In particular, σ_i is d-dimensional for C_h and O_h and for all $i \in I$.

For O_h one can verify easily that all results in this section remain true, except that Dirichlet boundary conditions are not satisfied. Moreover, Condition (2.5) can even be omitted; the operator is defined for all $u \in L^1(\Omega)$. Therefore Estimates (2.7), (2.8), (2.10), and (2.11) hold for $\ell = 0$ as well. Example 1 can be modified in the obvious way. (Z_h has to be substituted by O_h in all relations.)

For the Clement operator C_h, one has to decide whether Π_{σ_i} should be defined as in (1.6) or (1.8). In both cases the same estimates as for O_h can be proved. Note that we used in the proof only $\mathrm{C}_h w = w$ for $w \in \mathcal{P}_k^d$ which is satisfied. As discussed already in the Introduction, $\mathrm{C}_h v_h = v_h$ is in general not satisfied for $v_h \in V_h$.

Siebert [26] and Kunert [19] derived also some results for the operator C_h for anisotropic meshes. However, they considered only the case $k = 1$, $p = 2$, and only subsets $H^1_{\mathcal{T}}(\Omega) \subset W^{1,2}(\Omega)$ of so-called mesh adapted functions. This allows them to prove global results of the form

$$
\begin{aligned}
\sum_e \varrho_e^{-1} \| v - \mathrm{C}_h v, L^2(e) \| &\lesssim |v; W^{1,2}(\Omega)|, \\
\sum_e h_{i,e} \varrho_e^{-1} \left\| \frac{\partial}{\partial x_i} (v - \mathrm{C}_h v)), L^2(e) \right\| &\lesssim |v; W^{1,2}(\Omega)|, \quad i = 1, \ldots, d,
\end{aligned}
$$

where $\varrho_e \sim \min_{j=1,\ldots,d} h_{j,e}$. Using these estimates they prove asymptotic properties of a-posteriori error estimators. For v they insert the (exact) finite element error $u - u_h$. Unfortunately, the condition $u - u_h \in H^1_{\mathcal{T}}(\Omega)$ can not be proved/tested in general.

To satisfy Dirichlet boundary conditions all the authors [16, 19, 26] considered modifications of C_h near the boundary which is small enough to keep the approximation order.

3 The operator S_h: A modification of Z_h by choosing small sides

3.1 Stability and approximation in classical Sobolev spaces

Example 1 showed that anisotropic estimates of type $(1, \ell)$ are not valid for the Scott-Zhang operator in its general form. But for this example the following points were essential:

1. *Long edges* are chosen for σ_i.

2. X_i and X_j have the same x_1-coordinate but the projections of σ_i and σ_j on the x_1-axis are *different*.

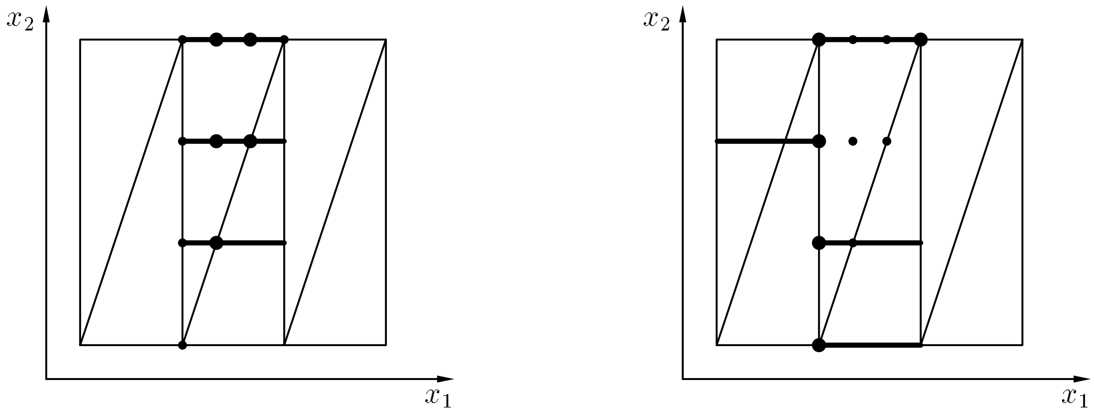
Because we have some freedom in the choice of σ_i we will investigate the operator in the cases where one of these points is avoided. In this section we will use short edges (2D) or small faces (3D) as σ_i. Large sides with identical projection are chosen in Section 4. The resulting operators will be denoted by S_h (small sides) and L_h (large sides).

Because the definition of the σ_i is different from that in Section 2 we will clarify this here: σ_i is (not necessarily uniquely) determined according to the following three properties, compare Figure 5.

(P1) σ_i is parallel to the x_1-axis/x_1, x_2-plane.

(P2) $X_i \in \overline{\sigma_i}$.

(P3) There exists a face ς of some element e such that the projection of ς on the x_1-axis/x_1, x_2-plane is identical with the projection of σ_i.



(a) Points where σ_i is uniquely determined. (b) Points where σ_i can be chosen (here one choice).

Figure 5: Choice of σ_i in dependence of X_i in the case of operator S_h, $k = 3$.

In connection with (P3) we have to note that σ_i is not necessary an edge/face of one element, see also Figure 5. Nevertheless, σ_i together with $\mathcal{P}_k^{d-1}$ or $(\mathcal{P}_k^1)^{d-1}$ is a Lagrangian finite element of dimension $d-1$, which follows from the tensor-product character of the elements e. For simplicity, we will use the terminology "σ_i is an edge/face". We remark in particular that in the case of simplicial elements and $k \geq 2$ there is no d-dimensional finite element $e' \subset S_e$ such that $\sigma_i \subset \overline{e'}$. This implies that $\mathcal{P}_{k,\sigma_i} \neq V_h|_{\sigma_i}$ and in general $\Pi_{\sigma_i} v_h \neq v_h|_{\sigma_i}$ for $v_h \in V_h$. That means that we lose Property 5 of Page 2. However, we need in the proofs only $\Pi_{\sigma_i} w = w$ for $w \in \mathcal{P}_{k,\sigma_i}$ which is of course satisfied.

Because σ_i is said to be a *short* edge/face this implies

$$h_j \leq h_d \quad \text{in } S_e \quad (j = 1, \ldots, d). \tag{3.1}$$

Note that in three dimensions and according to (1.10), (1.11), only elements with $h_1 \sim h_2 \lesssim h_3$ can be treated. But this is sufficient to handle edge singularities, see Section 6.

We will see that for the operator S_h anisotropic estimates of type (m, ℓ), $m < \ell \leq k+1$, can be derived. The main difficulty is to prove the stability estimate. The approximation property follows then easily using Lemma 1 from Page 8. To elucidate the different techniques for derivatives in x_1- and x_d-direction we first formulate and prove two lemmata. Then we establish the main theorem of this section. Finally, we give an example which shows that estimates of type (m, m), $1 \leq m \leq k+1$, are impossible.

Lemma 4 *The derivative in x_d-direction satisfies an $(1,1)$-estimate. The relation*

$$\left\| \frac{\partial}{\partial x_d} \mathrm{S}_h u; L^q(e) \right\| \lesssim (\operatorname{meas} e)^{1/q - 1/p} \, |u; W^{1,p}(S_e)|$$

holds for $u \in W^{1,p}(S_e)$ and all $p, q \in [1, \infty]$.

Proof Using the definition of the operator S_h, compare (2.4), the Hölder inequality, Estimate (2.12), and the trace theorem (2.14), we obtain for all $w \in \mathcal{P}_0^d$

$$\begin{aligned} \left\| \frac{\partial}{\partial x_d} \mathrm{S}_h u; L^q(e) \right\| &= \left\| \frac{\partial}{\partial x_d} \mathrm{S}_h (u - w); L^q(e) \right\| \\ &\leq \sum_{i \in I_e} \left\| \frac{\partial \varphi_i}{\partial x_d}; L^q(e) \right\| \left| \int_{\sigma_i} (u - w) \psi_i \right| \end{aligned}$$

$$
\begin{aligned}
&\lesssim h_d^{-1}(\operatorname{meas} e)^{1/q} \sum_{i\in I_e} \|u-w; L^1(\sigma_i)\|\, \|\psi_i; L^\infty(\sigma_i)\| \\
&\lesssim h_d^{-1}(\operatorname{meas} e)^{1/q} \sum_{i\in I_e} (\operatorname{meas}\sigma_i)(\operatorname{meas} e)^{-1/p} \sum_{|\alpha|\le 1} h^\alpha \|D^\alpha(u-w); L^p(S_e)\|(\operatorname{meas}\sigma_i)^{-1} \\
&\lesssim h_d^{-1}(\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha|\le 1} h^\alpha \|D^\alpha(u-w); L^p(S_e)\|.
\end{aligned}
$$

Using Lemma 1 with $m=0$, $\ell=1$, and relying on (3.1) we obtain the assertion. □

Lemma 5 *The derivative in x_1-direction satisfies an $(1,2)$-estimate. The relation*

$$
\left\| \frac{\partial}{\partial x_1} \mathrm{S}_h u; L^q(e) \right\| \lesssim (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha|\le 1} h^\alpha |D^\alpha u; W^{1,p}(S_e)|
$$

holds for $u\in W^{2,p}(S_e)$ and all $p,q\in[1,\infty]$.

Proof Let $w=w(x_d)\in\mathcal{P}_k^1$. Then we get in analogy to the proof of Lemma 4

$$
\left\| \frac{\partial}{\partial x_1} \mathrm{S}_h u; L^q(e) \right\| \lesssim h_1^{-1}(\operatorname{meas} e)^{1/q}(\operatorname{meas}\sigma_i)^{-1} \sum_{i\in I_e} \|u-w; L^1(\sigma_i)\|.
$$

Introduce now $k+1$ (simply connected) $(d-1)$-dimensional domains $\zeta_j\subset S_e$ such that for all σ_i $(i\in I_e)$ there exists a $\zeta_j\supset\sigma_i$. Note that ζ_j $(j=0,\ldots,k)$ is isotropic with a diameter of order h_1. Consequently, we obtain

$$
\begin{aligned}
\left\| \frac{\partial}{\partial x_1} \mathrm{S}_h u; L^q(e) \right\| &\lesssim h_1^{-1}(\operatorname{meas} e)^{1/q}(\operatorname{meas}\sigma_i)^{-1} \sum_{j=0}^{k} \|u-w; L^1(\zeta_j)\| \\
&\le h_1^{-1}(\operatorname{meas} e)^{1/q}(\operatorname{meas}\sigma_i)^{-1} \sum_{j=0}^{k} \sum_{\substack{|\alpha|\le 1\\ \alpha_d=0}} h^\alpha \|D^\alpha(u-w); L^1(\zeta_j)\|.
\end{aligned}
$$

Observe now that $w=w_j=\text{const.}$ on ζ_j. On the other hand, because the ζ_j have different x_d-coordinate, we can define w from given w_j $(j=0,\ldots,k)$. So we can use the $(d-1)$-dimensional analogon of Lemma 1 to choose $w_j\in\mathcal{P}_0^{d-1}$ such that

$$
\sum_{\substack{|\alpha|\le 1\\ \alpha_d=0}} h^\alpha \|D^\alpha(u-w_j); L^1(\zeta_j)\| \lesssim \sum_{\substack{|\alpha|=1\\ \alpha_d=0}} h^\alpha \|D^\alpha u; L^1(\zeta_j)\|
$$

and to conclude with the trace theorem (2.14) (applied for each ζ_j)

$$
\begin{aligned}
\left\| \frac{\partial}{\partial x_1} \mathrm{S}_h u; L^q(e) \right\| &\lesssim (\operatorname{meas} e)^{1/q}(\operatorname{meas}\sigma_i)^{-1} \sum_{j=0}^{k} \sum_{\substack{|\alpha|=1\\ \alpha_d=0}} \|D^\alpha u; L^1(\zeta_j)\| && (3.2)\\
&\lesssim (\operatorname{meas} e)^{1/q-1/p} \sum_{\substack{|\alpha|=1\\ \alpha_d=0}} \sum_{|\beta|\le 1} h^\beta \|D^{\alpha+\beta} u; L^p(S_e)\|. && (3.3)
\end{aligned}
$$

Thus the proposition is proved. □

By analogy we can treat the derivative with respect to x_2 in the three-dimensional case.

Theorem 6 *Assume that $h_j \le h_d$ $(j = 0, \ldots, d)$. Then the modified Scott-Zhang operator S_h satisfies on anisotropic meshes of tensor-product type the following estimates of type (m, ℓ):*

$$|\mathrm{S}_h u; W^{m,q}(e)| \lesssim (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha| \le \ell - m} h^\alpha |D^\alpha u; W^{m,p}(S_e)|, \tag{3.4}$$

$$|u - \mathrm{S}_h u; W^{m,q}(e)| \lesssim (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha| = \ell - m} h^\alpha |D^\alpha u; W^{m,p}(S_e)|, \tag{3.5}$$

$0 \le m \le \ell - 1 \le k$, provided that $u \in W^{\ell,p}(S_e)$. For (3.5) the numbers $p, q \in [1, \infty]$ must be such that $W^{\ell,p}(e) \hookrightarrow W^{m,q}(e)$. For $m \ge 2$ we exclude triangular and tetrahedral elements.

Proof Consider first the stability estimate (3.4). For $m = 0$, (3.4) can be proved as (2.10). For $m = 1$, (3.4) is proved in Lemmata 4 and 5. Let $m \ge 2$. Consider a multi-index γ with $|\gamma| = m$ and define $m_2 := \gamma_d$, $m_1 = m - m_2$. For arbitrary $\omega_1 \in \mathcal{P}^{d-1}_{m_1-1} \times \mathcal{P}^1_k$ (that is why we exclude simplicial elements) and $\omega_2 \in \mathcal{P}^d_{m-1}$ we obtain in analogy to the proof of Lemma 5

$$\begin{aligned}
\|D^\gamma \mathrm{S}_h u; L^q(e)\| &= \|D^\gamma \mathrm{S}_h((u - \omega_2) - \omega_1); L^q(e)\| \\
&\lesssim h^{-\gamma} (\operatorname{meas} e)^{1/q} (\operatorname{meas} \sigma_i)^{-1} \sum_{i \in I_e} \|u - \omega_2 - \omega_1; L^1(\sigma_i)\| \\
&\lesssim h^{-\gamma} (\operatorname{meas} e)^{1/q} (\operatorname{meas} \sigma_i)^{-1} \sum_{j=0}^{k} \sum_{\substack{|\alpha| \le m_1 \\ \alpha_d = 0}} h^\alpha \|D^\alpha (u - \omega_2 - \omega_1); L^1(\zeta_j)\|.
\end{aligned}$$

Then we determine $w_j \in \mathcal{P}^{d-1}_{m_1-1}$ $(j = 0, \ldots, k)$ such that

$$\sum_{\substack{|\alpha| \le m_1 \\ \alpha_d = 0}} h^\alpha \|D^\alpha (u - \omega_2 - w_j); L^1(\zeta_j)\| \lesssim \sum_{\substack{|\alpha| = m_1 \\ \alpha_d = 0}} h^\alpha \|D^\alpha (u - \omega_2); L^1(\zeta_j)\|.$$

Note that the w_j depend on $(u - \omega_2)$ and ω_2 is still to be chosen. The polynomial ω_1 is now determined by the w_j $(j = 0, \ldots, k)$ such that the estimate can be continued by

$$\|D^\gamma \mathrm{S}_h u; L^q(e)\| \lesssim h_d^{-m_2} (\operatorname{meas} e)^{1/q} (\operatorname{meas} \sigma_i)^{-1} \sum_{j=0}^{k} \sum_{\substack{|\alpha| = m_1 \\ \alpha_d = 0}} \|D^\alpha (u - \omega_2); L^1(\zeta_j)\|. \tag{3.6}$$

Thus the factor $h_1^{-m_1}$ is eliminated. We proceed now as in the proof of Lemma 4. Using the trace theorem (2.14) for all j, α and with $\ell - m_1 \ge \ell - m \ge 1$ instead of ℓ we conclude

$$\begin{aligned}
\|D^\gamma \mathrm{S}_h u; L^q(e)\| &\lesssim h_d^{-m_2} (\operatorname{meas} e)^{1/q-1/p} \sum_{\substack{|\alpha| = m_1 \\ \alpha_d = 0}} \sum_{|\beta| \le \ell - m_1} h^\beta \|D^{\alpha+\beta}(u - \omega_2); L^p(S_e)\| \\
&\lesssim h_d^{-m_2} (\operatorname{meas} e)^{1/q-1/p} \sum_{|\delta| \le \ell - m} \sum_{|\beta| \le m_2} h^{\beta+\delta} |D^{\beta+\delta}(u - \omega_2); W^{m_1,p}(S_e)|.
\end{aligned}$$

Using Corollary 2 (Page 8) we obtain

$$\begin{aligned}
\|D^\gamma \mathrm{S}_h u; L^q(e)\| &\lesssim h_d^{-m_2} (\operatorname{meas} e)^{1/q-1/p} \sum_{|\delta| \le \ell - m} \sum_{|\beta| = m_2} h^{\beta+\delta} |D^{\beta+\delta} u; W^{m_1,p}(S_e)| \\
&\lesssim (\operatorname{meas} e)^{1/q-1/p} \sum_{|\delta| \le \ell - m} h^\delta |D^\delta u; W^{m,p}(S_e)|.
\end{aligned}$$

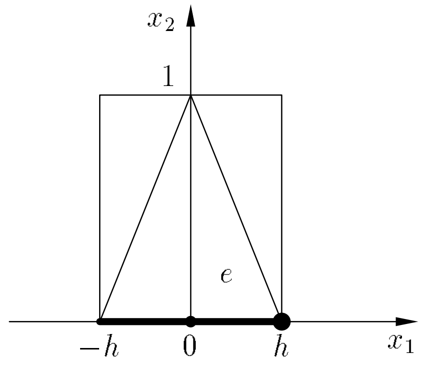


Figure 6: Illustration of Example 2.

Thus (3.4) is proved. Estimate (3.5) is a consequence of (3.4): For all $w \in \mathcal{P}^d_{\ell-1}$ we have

$$\begin{aligned} |u - \mathrm{S}_h u; W^{m,q}(e)| &\leq |u-w; W^{m,q}(e)| + |\mathrm{S}_h(u-w); W^{m,q}(e)| \\ &\lesssim (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha|\leq \ell-m} h^\alpha \|D^\alpha(u-w); W^{m,p}(S_e)\| \end{aligned}$$

and with Lemma 1 the proposition is proved. $\square$

Finally, we want to give an example that

$$|\mathrm{S}_h u; W^{1,2}(e)| \lesssim \|u; W^{1,2}(S_e)\| \tag{3.7}$$

does not hold for general $u \in W^{1,2}(S_e)$.

Example 2 Consider $k=1$ and a triangle with the vertices $X_1 = (0,0)$, $X_2 = (h,0)$, and $X_3 = (0,1)$, and let $\sigma_1 = (-h,0)\times\{0\}$, $\sigma_2 = (0,h)\times\{0\}$, compare Figure 6. For $u = r^\varepsilon \sin\frac{\theta}{2}$ (r,θ are here polar coordinates) we obtain

$$\begin{aligned} u|_{\sigma_1} = |x_1|^\varepsilon \;&\Rightarrow\; (\Pi_{\sigma_1}u)(X_1) = \int_0^h x^\varepsilon\left(-\frac{6x}{h^2}+\frac{4}{h}\right) \sim h^\varepsilon, \\ u|_{\sigma_2} = 0 \;&\Rightarrow\; (\Pi_{\sigma_2}u)(X_2) = 0. \end{aligned}$$

Consequently,

$$\frac{\partial \mathrm{S}_h u}{\partial x_1} \sim h^{\varepsilon-1}, \qquad |\mathrm{S}_h u; W^{1,2}(e)| \gtrsim h^{\varepsilon-1}(\operatorname{meas} e)^{1/2} = h^{\varepsilon-1/2} \to \infty$$

for $h \to 0$, $\varepsilon < \frac{1}{2}$. But

$$|u; W^{1,2}(S_e)|^2 \leq \int_0^1\int_0^\pi (r^{\varepsilon-1}\sin\tfrac{\theta}{2})^2\, r\,\mathrm{d}\theta\mathrm{d}r \sim \int_0^1 \mathrm{r}^{2(\varepsilon-1)+1}\,\mathrm{dr} < \infty$$

for $\varepsilon > 0$. Thus (3.7) does not hold.

3.2 Stability in weighted Sobolev spaces

We have seen in Example 2 that $\mathrm{S}_h u$ does not satisfy an estimate of type (1,1). However, S_h can be applied in some situations where $u \notin W^{2,p}(S_e)$ for some p we are interested in.

We restrict ourselves to the three-dimensional case, consider an arbitrary domain $G \subset \mathbb{R}^3$ and introduce cylindrical coordinates via $x_1 = r\cos\theta$, $x_2 = r\sin\theta$. Define for $\ell \in \mathbb{N}_0$, $p \in [1,\infty]$, $\beta \in \mathbb{R}$, the weighted Sobolev space

$$V_\beta^{\ell,p}(G) := \{v \in \mathcal{D}'(G) : \|v; V_\beta^{\ell,p}(G)\| < \infty\}, \tag{3.8}$$

$$\|v; V_\beta^{\ell,p}(G)\|^p := \sum_{|\alpha|\le\ell} \int_G |r^{\beta-\ell+|\alpha|} D^\alpha v|^p. \tag{3.9}$$

Such spaces are relevant in the treatment of singular functions of the type $v = r^\lambda \sin\lambda\theta$ or $v = r^\lambda \cos\lambda\theta$, $\lambda \in (0,1)$. Notice that

$$\begin{aligned} v \in W^{s,2}(G) \qquad\qquad &\iff s < 1+\lambda, \\ v \in V_\beta^{s,2}(G) \quad \forall s \ge 0 &\iff \beta > s-1-\lambda. \end{aligned}$$

For our application in Section 6 we need the stability of the modified Scott-Zhang operator in these weighted spaces.

Lemma 7 *Let m be an integer and β, p, q be real numbers with $0 \le m \le k$, $\beta < 2 - \frac{2}{p}$, $\beta \le 1$, $p, q \in [1,\infty]$, and assume that the x_3-axis proceeds through S_e. Then for $u \in W^{m,p}(S_e) \cap V_\beta^{m+1,p}(S_e)$ the stability estimate*

$$|\mathrm{S}_h u; W^{m,q}(e)| \le (\operatorname{meas} e)^{1/q-1/p} h_1^{-\beta} \sum_{|\alpha|=m-1} \sum_{|t|=1} h^t \|D^{\alpha+t} u; V_\beta^{1,p}(S_e)\| \tag{3.10}$$

holds. For $m \ge 2$ we exclude tetrahedral elements.

Proof We start with Estimate (3.6) which was obtained in the proof of Theorem 6. Let γ be a multi-index with $|\gamma| = m$, $m_1 = m - \gamma_3$, and $\omega_2 \in \mathcal{P}_{m-1}^d$. Then there holds

$$\|D^\gamma \mathrm{S}_h u; L^q(e)\| \lesssim h_3^{-\gamma_3} (\operatorname{meas} e)^{1/q} (\operatorname{meas}\sigma_i)^{-1} \sum_{j=0}^{k} \sum_{\substack{|\alpha|=m-\gamma_3 \\ \alpha_3=0}} \|D^\alpha(u-\omega_2); L^1(\zeta_j)\|. \tag{3.11}$$

Let $\gamma_3 > 0$, then we can continue, similar to the proof of Theorem 6, with the trace theorem because we assumed $u \in W^{m,p}(S_e)$.

$$\|D^\gamma \mathrm{S}_h u; L^q(e)\| \le h_3^{-\gamma_3} (\operatorname{meas} e)^{1/q-1/p} \sum_{\substack{|\alpha|=m-\gamma_3 \\ \alpha_3=0}} \sum_{|\delta|\le\gamma_3} h^\delta \|D^{\alpha+\delta}(u-\omega_2); L^p(S_e)\|.$$

Using Corollary 2 we obtain

$$\begin{aligned} \|D^\gamma \mathrm{S}_h u; L^q(e)\| &\le h_3^{-\gamma_3} (\operatorname{meas} e)^{1/q-1/p} \sum_{\substack{|\alpha|=m-\gamma_3 \\ \alpha_3=0}} \sum_{|\delta|=\gamma_3} h^\delta \|D^{\alpha+\delta} u; L^p(S_e)\| \\ &\le (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha|=m} \|D^\alpha u; L^p(S_e)\| \end{aligned} \tag{3.12}$$

We estimate the right hand side via the trivial embeddings $V^{1,p}_\beta(S_e) \hookrightarrow V^{0,p}_{\beta-1}(S_e) \hookrightarrow L^p(S_e)$, $\beta \le 1$, which leads with (3.1) to

$$\begin{aligned}
\sum_{|\alpha|=m} \|D^\alpha u; L^p(S_e)\| &\sim \sum_{|\alpha|=m-1} \sum_{|t|=1} \|D^{\alpha+t} u; L^p(S_e)\| \\
&\lesssim h_1^{-\beta+1} \sum_{|\alpha|=m-1} \sum_{|t|=1} \|r^{\beta-1} D^{\alpha+t} u; L^p(S_e)\| \\
&\lesssim h_1^{-\beta} \sum_{|\alpha|=m-1} \sum_{|t|=1} h^t \|D^{\alpha+t} u; V^{1,p}_\beta(S_e)\|, \qquad (3.13)
\end{aligned}$$

which is the desired result.

For $\gamma_3 = 0$ we use (3.11) with $\omega_2 = 0$ and estimate the $L^1(\zeta_j)$-norms against weighted norms via the Hölder inequality:

$$\|v; L^1(\zeta_j)\| \le \|r^{-\beta}; L^{p'}(\zeta_j)\| \cdot \|r^\beta v; L^p(\zeta_j)\| \qquad (3.14)$$

with p' from $\frac{1}{p} + \frac{1}{p'} = 1$. The $L^{p'}(\zeta_j)$-norm of $r^{-\beta}$ is finite if and only if $p'\beta < 2$ which is equivalent to $\beta < 2 - \frac{2}{p}$. Using $\operatorname{meas}\sigma_i \sim \operatorname{meas}\zeta_j \sim h_1^2$ for all i and j, and $r \lesssim h_1$ we get

$$\|r^{-\beta}; L^{p'}(\zeta_j)\| \lesssim h_1^{(-\beta p'+2)/p'} \sim (\operatorname{meas}\sigma_i)^{1-1/p} h_1^{-\beta}. \qquad (3.15)$$

The application of $W^{1,p}(S_e) \hookrightarrow L^p(\zeta_j)$ to $r^\beta v$ implies the trace theorem $V^{1,p}_\beta(S_e) \hookrightarrow V^{0,p}_\beta(\zeta_j)$ which leads to

$$\|r^\beta v; L^p(\zeta_j)\| \lesssim (\operatorname{meas}\sigma_i)^{1/p} (\operatorname{meas} e)^{-1/p} \sum_{|s|\le 1} h_1^{1-|s|} h^s \|r^{\beta-1+|s|} D^s v; L^p(S_e)\|.$$

Combining these estimates we obtain

$$\|v; L^1(\zeta_j)\| \le \operatorname{meas}\sigma_i \, (\operatorname{meas} e)^{-1/p} h_1^{-\beta} \sum_{|s|\le 1} h_1^{1-|s|} h^s \|r^{\beta-1+|s|} D^s v; L^p(S_e)\|$$

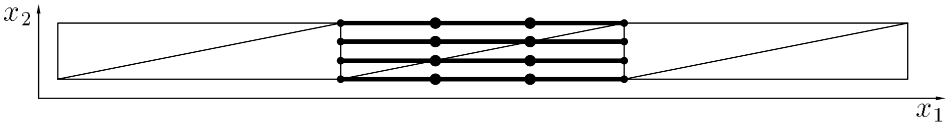
and thus with (3.11)

$$\begin{aligned}
\|D^\gamma \mathrm{S}_h u; L^q(e)\| &\lesssim (\operatorname{meas} e)^{1/q} (\operatorname{meas}\sigma_i)^{-1} \sum_{j=0}^{k} \sum_{|\alpha|=m} \|D^\alpha u; L^1(\zeta_j)\| \\
&\lesssim (\operatorname{meas} e)^{1/q-1/p} h_1^{-\beta} \sum_{|\alpha|=m} \sum_{|s|\le 1} h_1^{1-|s|} h^s \|r^\beta D^{\alpha+s} u; L^p(S_e)\|.
\end{aligned}$$

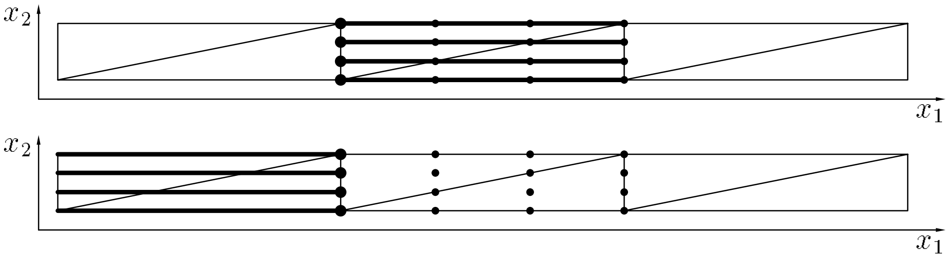
The last step to derive (3.10) is done in analogy to (3.13) using

$$\begin{aligned}
&\sum_{|t|=1} \sum_{|s|\le 1} h_1^{1-|s|} h^s \|r^{\beta-1+|s|} D^{t+s} u; L^p(S_e)\| \\
&\quad = \sum_{|t|=1} \sum_{|s|=1} h^s \|r^\beta D^{t+s} u; L^p(S_e)\| + \sum_{|t|=1} h_1 \|r^{\beta-1} D^t u; L^p(S_e)\| \\
&\quad \lesssim \sum_{|t|=1} \sum_{|s|=1} h^s \|r^\beta D^{t+s} u; L^p(S_e)\| + \sum_{|s|=1} h^s \|r^{\beta-1} D^s u; L^p(S_e)\| \\
&\quad \sim \sum_{|s|=1} h^s \|D^s u; V^{1,p}_\beta(S_e)\|.
\end{aligned}$$

□



(a) Points where σ_i is uniquely determined.



(b) Two choices for σ_i for points on vertical mesh lines.

Figure 7: Choice of σ_i in dependence of X_i in the case of operator L_h.

4 The operator L_h: A modification of Z_h by choosing long sides with a projection property

In contrast to Section 3 we will now employ large edges/faces and denote the resulting operator by L_h. The notation is used as follows: We keep Properties (P1), (P2), and (P3) from Page 10 and simply turn the relation (3.1):

$$h_j \geq h_d \quad \text{in } S_e \quad (j = 1, \ldots, d). \tag{4.1}$$

But in correspondence with Item 2 at the beginning of Section 3, we do not have so much freedom for the choice of the σ_i as in the case of S_h. We must assume the following projection property (P4), compare also Figure 7.

(P4) If the projections of any two points X_i and X_j on the x_1-axis/x_1, x_2-plane coincide then so do the projections of σ_i and σ_j.

We can prove the results of Theorem 6 for this case as well. Moreover, these results extend to the case $m = \ell$. But in contrast to the needle elements of Section 3 the three-dimensional elements are now flat, $h_1 \sim h_2 \gtrsim h_3$. The idea for this choice of σ_i was found in [12, Chapter 5] where the special case of rectangular and brick elements was considered for $k = 1$, $p = q = 2$. We extend this theory to more element types and to general $k \in \mathbb{N}$, $p, q \in [1, \infty]$. Our proof differs from that in [12].

We start as in Section 3 with the separate consideration of the stability of first derivatives of $\mathrm{L}_h u$. This time the derivative in x_1-direction is the simpler one.

Lemma 8 *The estimate of type* $(1,1)$

$$\left\| \frac{\partial}{\partial x_n} \mathrm{L}_h u; L^q(e) \right\| \leq (\operatorname{meas} e)^{1/q - 1/p} |u; W^{1,p}(S_e)|, \quad n = 1, \ldots, d. \tag{4.2}$$

holds.

Proof For $n = 1, \ldots, d-1$ the proof can be carried out with the same arguments as the proof of Lemma 4. The only difference is that the role of x_d and h_d is now played by x_n and h_n.

For the case $n = d$ we will reformulate $\mathrm{L}_h u$. For this consider first a one-dimensional situation, that means a single finite element formed by an interval (ξ, η). Let ϕ_i, $i = 0, \dots, k$, be the nodal basis functions in (ξ, η). We change now to a new basis

$$\chi_i = \sum_{j=0}^{i} \phi_j, \quad i = 0, \dots, k.$$

Consequently,

$$\sum_{i=0}^{k} a_i \phi_i = \sum_{i=0}^{k-1} (a_i - a_{i+1}) \chi_i + a_k,$$

where we also used that $\sum_{i=0}^{k} \phi_i = 1$. Note further that

$$\|\chi_i; L^\infty(\xi, \eta)\| \lesssim 1, \quad \|\chi_i'; L^\infty(\xi, \eta)\| \lesssim |\eta - \xi|^{-1}. \tag{4.3}$$

We use this kind of a new basis in the case of a rectangular element $e = (\xi_1, \eta_1) \times (\xi_2, \eta_2)$. The nodal basis functions are (for simplicity with a double index)

$$\varphi_{i,j}(x_1, x_2) = \phi^i(x_1) \phi_j(x_2), \quad i, j = 0, \dots, k, \tag{4.4}$$

where ϕ^i and ϕ_j are the nodal basis functions with respect to (ξ_1, η_1) and (ξ_2, η_2), respectively. Thus

$$\begin{aligned}
\mathrm{L}_h u &= \sum_{i=0}^{k} \sum_{j=0}^{k} a_{i,j} \phi^i(x_1) \phi_j(x_2) \\
&= \sum_{i=0}^{k} \phi^i(x_1) \left(\sum_{j=0}^{k-1} (a_{i,j} - a_{i,j+1}) \chi_j(x_2) + a_{i,k} \right), \\
\frac{\partial}{\partial x_2} \mathrm{L}_h u &= \sum_{i=0}^{k} \phi^i(x_1) \sum_{j=0}^{k-1} (a_{i,j} - a_{i,j+1}) \chi_j'(x_2).
\end{aligned} \tag{4.5}$$

Because of Property (P4) the subdomains $\sigma_{i,j}$ belonging to the node (i,j) depend only on i. We can write

$$\begin{aligned}
a_{i,j} &= \int_{\sigma_{i,j}} \psi_i(x_1) u(x_1, y_j) \, \mathrm{d}x_1, \\
a_{i,j} - a_{i,j+1} &= -\int_{\sigma_{i,j}} \psi_i(x_1) \int_{y_j}^{y_j+1} \frac{\partial u}{\partial x_2}(x_1, y) \, \mathrm{d}y \mathrm{d}x_1, \\
\sum_{j=0}^{k-1} |a_{i,j} - a_{i,j+1}| &\leq \int_{S_e} \left| \psi_i \frac{\partial u}{\partial x_2} \right|,
\end{aligned} \tag{4.6}$$

where y_j is the value of the x_2-coordinate of points $X_{i,j}$. The proof of (4.2) is now standard:

$$\begin{aligned}
\left\| \frac{\partial}{\partial x_d} \mathrm{L}_h u; L^q(e) \right\| &\lesssim \sum_{i=0}^{k} \sum_{j=0}^{k-1} |a_{i,j} - a_{i,j+1}| \cdot \|\phi^i(x_1) \chi_j'(x_2); L^q(e)\| \\
&\lesssim h_2^{-1} (\operatorname{meas} e)^{1/q} \sum_{i=0}^{k} \int_{S_e} \left| \psi_i \frac{\partial u}{\partial x_2} \right| \\
&\lesssim h_2^{-1} (\operatorname{meas} e)^{1/q+1-1/p} \sum_{i=0}^{k} (\operatorname{meas} \sigma_i)^{-1} \left\| \frac{\partial u}{\partial x_2}; L^p(S_e) \right\|.
\end{aligned}$$

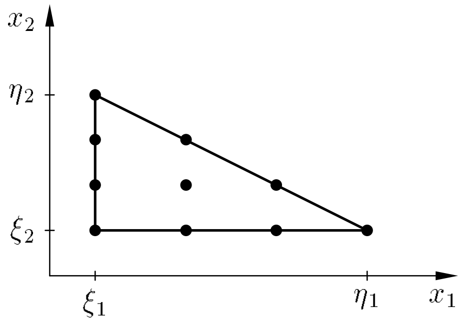


Figure 8: Illustration of the case of an triangle.

For pentahedral and hexahedral elements the proof is similar. We only replace (4.4) by

$$\varphi_{i,j}(x_1,x_2,x_3) = \phi^i(x_1,x_2)\phi_j(x_3), \quad i=0,\dots,K,\ j=0,\dots,k,$$

with appropriate basis functions $\phi^i(x_1,x_2)$ and

$$K = (k+1)^2 - 1 \quad \text{for hexahedra,} \quad K = \binom{k+2}{2} - 1 \quad \text{for pentahedra.} \tag{4.7}$$

In the case of simplicial elements we have to modify these considerations slightly. We will explain it in the two-dimensional case. Consider an element e with nodes $X_{i,j}$,

$$\begin{aligned} e &= \left\{ (x_1,x_2) : \xi_1 \le x_1 \le \eta_1,\ \xi_2 \le x_2 \le \eta_2 - (x_1-\xi_1)\frac{\eta_2-\xi_2}{\eta_1-\xi_1} \right\}, \\ X_{i,j} &= \left(\xi_1 + \frac{i}{k}(\eta_1-\xi_1), \xi_2 + \frac{j}{k}(\eta_2-\xi_2) \right), \end{aligned}$$

and nodal basis functions $\varphi_{i,j}$, $i=0,\dots,k$, $j=0,\dots,k-i$, as illustrated in Figure 8. The new basis functions are

$$\chi_{i,j} = \sum_{s=0}^{j} \varphi_{i,s}, \quad i=0,\dots,k,\ j=0,\dots,k-i.$$

We get

$$\begin{aligned} \mathrm{L}_h u &= \sum_{i=0}^{k}\sum_{j=0}^{k-i} a_{i,j}\varphi_{i,j} = \sum_{i=0}^{k}\left(\sum_{j=0}^{k-i-1} (a_{i,j}-a_{i,j+1})\chi_{i,j} + a_{i,k-i}\chi_{i,k-i} \right), \\ \left\| \frac{\partial \mathrm{L}_h u}{\partial x_2}; L^q(e) \right\| &\lesssim \sum_{i=0}^{k}\left(\sum_{j=0}^{k-i-1} |a_{i,j}-a_{i,j+1}| \left\| \frac{\partial \chi_{i,j}}{\partial x_2}; L^q(e) \right\| + |a_{i,k-i}| \left\| \frac{\partial \chi_{i,k-i}}{\partial x_2}; L^q(e) \right\| \right). \end{aligned}$$

To conclude (4.2) with the same arguments as above it remains to show that

$$\frac{\partial \chi_{i,k-i}}{\partial x_2} = 0 \quad \text{for all } i=0,\dots,k. \tag{4.8}$$

For this we observe that $\chi_{i,k-i}$ is uniquely determined by

$$\chi_{i,k-i}(X_{s,j}) = \begin{cases} 1 & \text{for } s=i,\ j=0,\dots,k-i, \\ 0 & \text{else.} \end{cases}$$

Thus $\chi_{i,k-i} = \phi^i(x_1)$ with ϕ^i in the sense of (4.4), and (4.8) is proved.

The proof for tetrahedral elements is analogous. $\square$

Theorem 9 *Assume that $h_j \geq h_d$ $(j = 0, \ldots, d)$. On anisotropic meshes of tensor-product type the modified Scott-Zhang operator L_h satisfies the following estimates:*

$$|\mathrm{L}_h u; W^{m,q}(e)| \lesssim (\operatorname{meas} e)^{1/q-1/p} |u; W^{m,p}(S_e)|, \tag{4.9}$$

$$|u - \mathrm{L}_h u; W^{m,q}(e)| \lesssim (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha|=\ell-m} h^\alpha |D^\alpha u; W^{m,p}(S_e)|, \tag{4.10}$$

$0 \leq m \leq \ell$, $1 \leq \ell \leq k+1$, *provided that* $u \in W^{\ell,p}(S_e)$. *For* (4.10) *the numbers* $p, q \in [1, \infty]$ *must be such that* $W^{\ell,p}(e) \hookrightarrow W^{m,q}(e)$.

Proof Estimate (4.10) follows from (4.9) via Lemma 1 as it was done for S_h in the proof of Theorem 6. So the main point is to prove (4.9). For $m = 0$, this can be done as in the proof of (2.10). The case $m = 1$ is treated in Lemma 8.

Let $m \geq 2$. Consider a multi-index γ with $|\gamma| = m$ and define $m_2 := \gamma_d$, $m_1 := m - m_2$. In the proof of Lemma 8, we made for the case $m_2 = 1$ a transformation of the nodal basis $\varphi_{i,j}$ to a basis $\chi_{i,j}$ in order to obtain differences of first order:

$$\frac{\partial}{\partial x_d} \sum_{i=0}^{K} \sum_{j=0}^{k} a_{i,j} \varphi_{i,j} = \frac{\partial}{\partial x_d} \sum_{i=0}^{K} \sum_{j=0}^{k-1} (a_{i,j} - a_{i,j+1}) \chi_{i,j}.$$

This process is repeated until differences of order m_2 are created: For simplicity consider again the one-dimensional situation. We define recursively coefficients $a_i^{(n)}$ and functions $\chi_i^{(n)}$, $i = 0, \ldots, k-n$, $n = 0, \ldots, m_2$, by

$$a_i^0 := a_i, \quad a_i^{(n+1)} := a_i^{(n)} - a_{i+1}^{(n)}, \quad i = 0, \ldots, k-n,$$

$$\chi_i^0 := \varphi_i, \quad \chi_i^{(n+1)} := \sum_{s=0}^{i} \chi_s^{(n)}, \qquad i = 0, \ldots, k,$$

and obtain

$$\frac{\partial^{m_2}}{\partial x^{m_2}} \sum_{i=0}^{k} a_i \varphi_i = \frac{\partial^{m_2}}{\partial x^{m_2}} \sum_{i=0}^{k-m_2} a_i^{(m_2)} \chi_i^{(m_2)}. \tag{4.11}$$

We get this by induction in analogy to the proof of Lemma 8. The only point is to prove that

$$\frac{\partial^{n+1}}{\partial x^{n+1}} \chi_{k-n}^{(n+1)} = 0 \quad \text{for } n = 0, \ldots, m_2 - 1.$$

This can be shown for any fixed n via $\chi_i^{(n+1)} = \sum_{s=0}^{i} \binom{i-s+n}{n} \chi_s^{(0)}$ (proof by induction) which yields $\chi_k^{(n+1)} = \sum_{s=0}^{k} \binom{k-s+n}{n} \varphi_s$, $\chi_k^{(n+1)}(X_r) = \binom{k-r+n}{n}$, $r = 0, \ldots, k$, $\chi_k^{(n+1)} \in \mathcal{P}_n^1$. From $\chi_i^{(n+1)} = \chi_{i+1}^{(n+1)} - \chi_{i+1}^{(n)}$ this gives by induction $\chi_i^{(n+1)} \in \mathcal{P}_n^1$ for $i = k, k-1, \ldots, k-n$. Thus $\frac{\partial^{n+1}}{\partial x^{n+1}} \chi_i^{(n+1)} = 0$ for $i = k-n, \ldots, k$.

Consider now rectangular elements ($d = 2$) and transfer this basis transformation to the x_2-direction. We derive (again by induction) from (4.11)

$$\frac{\partial^{m_2}}{\partial x_d^{m_2}} \sum_{i=0}^{k} \sum_{j=0}^{k} a_{i,j} \varphi_{i,j} = \frac{\partial^{m_2}}{\partial x_d^{m_2}} \sum_{i=0}^{k} \sum_{j=0}^{k-m_2} a_{i,j}^{(m_2)} \chi_{i,j}^{(m_2)}. \tag{4.12}$$

The so created differences $a_{i,j}^{(n+1)} = a_{i,j}^{(n)} - a_{i+1,j}^{(n)}$ are used now to establish an integral representation; compare (4.6):

$$a_{i,j}^{(1)} = -\int_{\sigma_{i,j}} \psi_i(x_1) \int_0^\delta \frac{\partial u}{\partial x_d}(x_1, y_j + \eta_1) \, \mathrm{d}\eta_1 \mathrm{d}x_1,$$

$\delta = y_{j+1} - y_j$ is assumed to be independent of j. We continue recursively and obtain

$$\begin{aligned}
a_{i,j}^{(2)} &= -\int_{\sigma_{i,j}} \psi_i(x_1) \left[\int_0^\delta \frac{\partial u}{\partial x_d}(x_1, y_j + \eta_1)\,\mathrm{d}\eta_1 - \int_0^\delta \frac{\partial u}{\partial x_d}(x_1, y_{j+1} + \eta_1)\,\mathrm{d}\eta_1 \right] \mathrm{d}x_1 \\
&= (-1)^2 \int_{\sigma_{i,j}} \psi_i(x_1) \int_0^\delta \int_0^\delta \frac{\partial^2 u}{\partial x_d^2}(x_1, y_j + \eta_1 + \eta_2)\,\mathrm{d}\eta_1 \mathrm{d}\eta_2 \mathrm{d}x_1, \\
a_{i,j}^{(n)} &= (-1)^n \int_{\sigma_{i,j}} \psi_i(x_1) \underbrace{\int_0^\delta \cdots \int_0^\delta}_{n \text{ times}} \frac{\partial^n u}{\partial x_d^n}(x_1, y_j + \eta_1 + \cdots + \eta_n)\,\mathrm{d}\eta_1 \cdots \mathrm{d}\eta_n \mathrm{d}x_1.
\end{aligned}$$

Using (2.12) and $\delta \sim h_2$ we obtain

$$|a_{i,j}^{(n)}| \lesssim (\operatorname{meas}\sigma_i)^{-1} h_d^{n-1} \left\| \frac{\partial^n u}{\partial x_d^n}; L^1(S_e) \right\|.$$

Replace now σ_i by $\sigma := \min_{i=0,\dots,k} \sigma_i$ and u by $u - w$, $w \in \mathcal{P}_{m-1}^2$ arbitrary. Together with (4.12) we conclude that

$$\begin{aligned}
\|D^\gamma \mathrm{L}_h u; L^q(e)\| &= \|D^\gamma \mathrm{L}_h (u - w); L^q(e)\| \\
&\lesssim \sum_{i=0}^{k} \sum_{j=0}^{k-m_2} |a_{i,j}^{(m_2)}| \|D^\gamma \chi_{i,j}^{(m_2)}; L^q(e)\| \\
&\lesssim h^{-\gamma} (\operatorname{meas} e)^{1/q} \sum_{i=0}^{k} \sum_{j=0}^{k-m_2} |a_{i,j}^{(m_2)}| \\
&\lesssim h^{-\gamma} (\operatorname{meas} e)^{1/q} (\operatorname{meas}\sigma)^{-1} h_d^{m_2-1} \left\| \frac{\partial^{m_2}}{\partial x_d^{m_2}}(u - w); L^1(S_e) \right\| \\
&\lesssim h^{-\gamma} h_d^{m_2} (\operatorname{meas} e)^{1/q-1/p} \left\| \frac{\partial^{m_2}}{\partial x_d^{m_2}}(u - w); L^p(S_e) \right\| \\
&\lesssim h^{-m_1} (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha| \le m-m_2} h^\alpha \left\| D^\alpha \frac{\partial^{m_2}}{\partial x_d^{m_2}}(u - w); L^p(S_e) \right\|.
\end{aligned} \tag{4.13}$$

Via Corollary 2, (4.1), and $m = m_1 + m_2$ we obtain

$$\begin{aligned}
\|D^\gamma \mathrm{L}_h u; L^q(e)\| &\lesssim h^{-m_1} (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha| = m-m_2} h^\alpha \left\| D^\alpha \frac{\partial^{m_2} u}{\partial x_d^{m_2}}; L^p(S_e) \right\| \\
&\le (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha| = m-m_2} \left\| D^\alpha \frac{\partial^{m_2} u}{\partial x_d^{m_2}}; L^p(S_e) \right\| \\
&\le (\operatorname{meas} e)^{1/q-1/p} |u; W^{m,p}(S_e)|
\end{aligned}$$

and (4.9) is proved for rectangular elements. The proof for all other types of elements is similar using the ideas explained in the proof of Lemma 8. □

5 The operator E_h: Choosing long edges in the three-dimensional case

5.1 Stability and approximation in Sobolev spaces

As already mentioned in Section 4 we will now investigate the general three-dimensional situation of independent mesh sizes h_1, h_2, and h_3. In order to obtain in Subsection 5.2 a notation which is compatible with that in Subsection 3.2 we let

$$h_1 \le h_2 \le h_3. \tag{5.1}$$

Assume, for simplicity, tensor product meshes in the sense that transformation (1.10) is reduced to

$$x_i = h_{i,e}\hat{x}_i, \quad (i = 1, 2, 3). \tag{5.2}$$

The investigation of the operators S_h and L_h was based on taking σ_i as isotropic faces, that means that h_2 is of the same order as h_1 or h_3. In [12] it was suggested to overcome this restriction by taking *one-dimensional* σ_i but this was not elaborated thoroughly. We will now investigate which estimates can be obtained in this case. We assume the following properties which are analogous to the ones in Section 4.

(P1′) σ_i is parallel to the x_3-axis.

(P2) $X_i \in \overline{\sigma_i}$.

(P3′) There exists an edge ς of some element e such that the projection of ς on the x_3-axis is identical with the projection of σ_i.

(P4′) If the projections of any two points X_i and X_j on the x_3-axis coincide then so do the projections of σ_i and σ_j.

The corresponding operator is denoted by $\mathrm{E}_h : W^{\ell,p}(\Omega) \to V_h$. Note that it is defined only for $u \in W^{\ell,p}(\Omega)$ with

$$\ell \ge 2 \quad \text{for } p = 1, \qquad \ell > \frac{2}{p} \quad \text{otherwise}, \tag{5.3}$$

to guarantee that $u|_{\sigma_i} \in L^1(\sigma_i)$. Condition (5.3) can be reformulated to

$$\ell \ge 2,\ p \in [1, \infty] \quad \text{or} \quad \ell = 1,\ p \in (2, \infty]. \tag{5.4}$$

Theorem 10 *Assume that* (5.1) *and* (5.2) *are fulfilled. Then the operator* E_h *satisfies for all* $q \in [1, \infty]$ *the following estimates:*

$$|\mathrm{E}_h u; W^{m,q}(e)| \lesssim (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha|\le 1} h^\alpha |D^\alpha u; W^{m,p}(S_e)| \tag{5.5}$$

if $m \ge 1$ *or* $p > 2$, *and*

$$\|\mathrm{E}_h u; L^q(e)\| \lesssim (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha|\le \ell} h^\alpha \|D^\alpha u; L^p(S_e)\| \tag{5.6}$$

with ℓ *and* p *satisfying (5.4). The approximation error estimate*

$$|u - \mathrm{E}_h u; W^{m,q}(e)| \lesssim (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha|=\ell-m} h^\alpha |D^\alpha u; W^{m,p}(S_e)| \tag{5.7}$$

holds if $0 \le m \le \ell - 1 \le k$, p *satisfies* (5.4), q *is such that* $W^{\ell,p}(e) \hookrightarrow W^{m,q}(e)$, *and* $u \in W^{\ell,p}(S_e)$.

We will see in the proof that for certain derivatives $D^\gamma \mathrm{E}_h u$ the stability estimate (5.5) can still be improved.

Proof We prove the theorem for brick elements. Other element types are treated similarly, see the discussion in the proof of Lemma 8. We have to consider different cases separately.

First, let γ be a multi-index with $|\gamma| = m$ and $\gamma_1 \neq 0$, $\gamma_2 \neq 0$. We use the difference technique developed in the proof of Theorem 9 for both directions x_1 and x_2. In analogy to (4.13) we obtain for all $w \in \mathcal{P}^3_{m-1}$

$$\begin{aligned}\|D^\gamma \mathrm{E}_h u, L^q(e)\| &= \|D^\gamma \mathrm{E}_h (u-w), L^q(e)\| \\ &\lesssim h^{-\gamma} h_1^{\gamma_1} h_2^{\gamma_2} (\operatorname{meas} e)^{1/q-1/p} \left\| \frac{\partial^{\gamma_1}}{\partial x_1^{\gamma_1}} \frac{\partial^{\gamma_2}}{\partial x_2^{\gamma_2}} (u-w); L^p(S_e) \right\| \\ &\leq h_3^{-\gamma_3} (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha|\leq\gamma_3} h^\alpha |D^\alpha u; W^{\gamma_1+\gamma_2,p}(S_e)|.\end{aligned}$$

Using Corollary 2 and (5.1) we conclude

$$\begin{aligned}\|D^\gamma \mathrm{E}_h u, L^q(e)\| &\lesssim h_3^{-\gamma_3} (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha|=\gamma_3} h^\alpha |D^\alpha u; W^{\gamma_1+\gamma_2,p}(S_e)| \\ &\leq (\operatorname{meas} e)^{1/q-1/p} |u; W^{m,p}(S_e)|.\end{aligned}$$

In a second case we assume $\gamma_n \neq 0$, $n = 1$ or $n = 2$, but $\gamma_{3-n} = 0$, $\gamma_3 \neq 0$. Then we can use the difference technique only within some faces f_i ($i = 0, \ldots, k$) which are parallel to the x_n, x_3-plane. Defining $f := \bigcup_{i=0}^k f_i$ we find as above that for all $w \in \mathcal{P}^3_{m-1}$

$$\begin{aligned}\|D^\gamma \mathrm{E}_h u, L^q(e)\| &= \|D^\gamma \mathrm{E}_h (u-w), L^q(e)\| \\ &\lesssim h^{-\gamma} h_n^{\gamma_n} (\operatorname{meas} e)^{1/q} (\operatorname{meas} f)^{-1/p} \left\| \frac{\partial^{\gamma_n}}{\partial x_n^{\gamma_n}} (u-w); L^p(f) \right\|. \qquad (5.8)\end{aligned}$$

Using the trace theorem $W^{\gamma_3,p}(S_e) \hookrightarrow L^p(f)$ and again Corollary 2 as well as (5.1) we obtain

$$\begin{aligned}\|D^\gamma \mathrm{E}_h u, L^q(e)\| &\lesssim h_3^{-\gamma_3} (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha|\leq\gamma_3} h^\alpha |D^\alpha (u-w); W^{\gamma_n,p}(S_e)| \\ &\lesssim h_3^{-\gamma_3} (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha|=\gamma_3} h^\alpha |D^\alpha u; W^{\gamma_n,p}(S_e)| \\ &\leq (\operatorname{meas} e)^{1/q-1/p} |u; W^{m,p}(S_e)|.\end{aligned}$$

Consider now the remaining pure derivatives. Let first be $\gamma_n = m$, $n = 1$ or $n = 2$, $\gamma_3 = 0$. Estimate (5.8) holds in this case as well. By using $p = 1$ and $w = 0$ it reads now

$$\|D^\gamma \mathrm{E}_h u, L^q(e)\| \lesssim (\operatorname{meas} e)^{1/q} (\operatorname{meas} f)^{-1} \|D^\gamma u; L^1(f)\|. \qquad (5.9)$$

With the trace theorem $W^{1,p}(S_e) \hookrightarrow L^1(f)$ for all $p \in [1, \infty]$ we conclude the assertion (5.5).

Finally, for $\gamma_3 = m$, $\gamma_1 = \gamma_2 = 0$, the proof of the stability is completely analogous to the proof of Lemma 4. We have for all $w \in \mathcal{P}^3_{m-1}$

$$\|D^\gamma \mathrm{E}_h u, L^q(e)\| \lesssim h_3^{-m} (\operatorname{meas} e)^{1/q} \sum_{i\in I_e} (\operatorname{meas} \sigma_i)^{-1} \|u-w; L^1(\sigma_i)\|.$$

The trace theorem $W^{m+1,p}(S_e) \hookrightarrow L^1(\sigma_i)$ (which is the reason for the assumption $m \geq 1$ or $p > 2$) and Corollary 2 yield

$$\|D^\gamma \mathrm{E}_h u, L^q(e)\| \lesssim h_3^{-m} (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha|\leq m} \sum_{|\beta|\leq 1} h^{\alpha+\beta} \|D^{\alpha+\beta}(u-w); L^p(S_e)\|$$

$$\begin{aligned}
&\lesssim h_3^{-m} (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha|=m} \sum_{|\beta|\le 1} h^{\alpha+\beta} \| D^{\alpha+\beta} u; L^p(S_e)\| \\
&\lesssim (\operatorname{meas} e)^{1/q-1/p} \sum_{|\beta|\le 1} h^{\beta} | D^{\beta} u; W^{m,p}(S_e)|.
\end{aligned}$$

Note that in this last case ($\gamma_3 = m$) for $m \geq 2$ and for $m = 1$, $p > 2$, it can even be proved that

$$\| D^\gamma \mathrm{E}_h u, L^q(e)\| \lesssim (\operatorname{meas} e)^{1/q-1/p} |u; W^{m,p}(S_e)|$$

because then $W^{m,p}(S_e) \hookrightarrow L^1(\sigma_i)$ holds.

Estimate (5.6) is trivial since

$$\| \mathrm{E}_h u, L^q(e)\| \lesssim (\operatorname{meas} e)^{1/q} \sum_{i \in I_e} (\operatorname{meas} \sigma_i)^{-1} \| u; L^1(\sigma_i)\|,$$

and the embedding $W^{\ell,p}(S_e) \hookrightarrow L^1(\sigma_i)$ holds just for ℓ, p satisfying (5.4).

Estimate (5.7) is concluded from (5.5) and (5.6) as in the proof of Theorem 6. □

It is interesting to point out that the proof shows that

$$\| D^\gamma \mathrm{E}_h u, L^q(e)\| \lesssim (\operatorname{meas} e)^{1/q-1/p} |u; W^{m,p}(S_e)| \tag{5.10}$$

holds for γ with $|\gamma| = m$ if at most one of the numbers $\gamma_1, \gamma_2, \gamma_3$ vanishes. Our way of proof does not work for pure derivatives. Consider for example the case $\gamma = (1, 0, 0)$. To prove (5.10) with $p > 2$ ($\mathrm{E}_h u$ is defined only for $u \in W^{1,p}(\Omega)$ with $p > 2$.) one would have to skip the trace on f and to use a trace theorem in the form (2.14). But this leads to

$$\| D^\gamma \mathrm{E}_h u, L^q(e)\| \lesssim h_1^{-1} (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha|\le 1} h^{\alpha} \| D^{\alpha} u; L^p(S_e)\|$$

with some diverging terms at the right hand side. The case $\gamma = (1, 0, 0)$ would be tractable only if

$$\| D^\gamma \mathrm{E}_h u, L^q(e)\| \lesssim (\operatorname{meas} e)^{1/q-1/p} \| D^\gamma u; L^p(S_e)\|$$

was valid. It is not clear whether this estimate holds.

Remark 1 Our motivation for introducing the operator E_h was to be able to treat the general case of three independent mesh sizes $h_1 \leq h_2 \leq h_3$. Of course this includes the special case $h_1 \sim h_2$. We point out that in this case the transformation (5.2) can be generalized to (1.10), (1.11). To see that then the statement of Theorem 10 is still true consider an arbitrary element $e \in \mathcal{T}_h$ and denote its projection into the x_1, x_2-plane by ζ. Because $\mathcal{T}_h$ is of tensor product type, and because all σ_i are perpendicular to the x_1, x_2-plane, it suffices to choose S_e such that its projection to the x_1, x_2-plane is again ζ (and $\sigma_i \subset \overline{S_e}$), compare Figure 9. Via the transformation

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} h_1^{-1} B_e & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} =: \tilde{B} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix},$$

B_e from (1.10), the domains e and S_e can be mapped to $\tilde{e}$ and $\tilde{S}_e = S_{\tilde{e}}$ which satisfy (locally) the assumptions made at the beginning of this section. That means that Theorem 10 holds true with respect to the coordinate system $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$. By observing that

$$\det \tilde{B} \sim 1, \quad \|\tilde{B}\| \sim 1, \quad \|\tilde{B}^{-1}\| \sim 1$$

we find that Theorem 10 extends to the meshes described above.

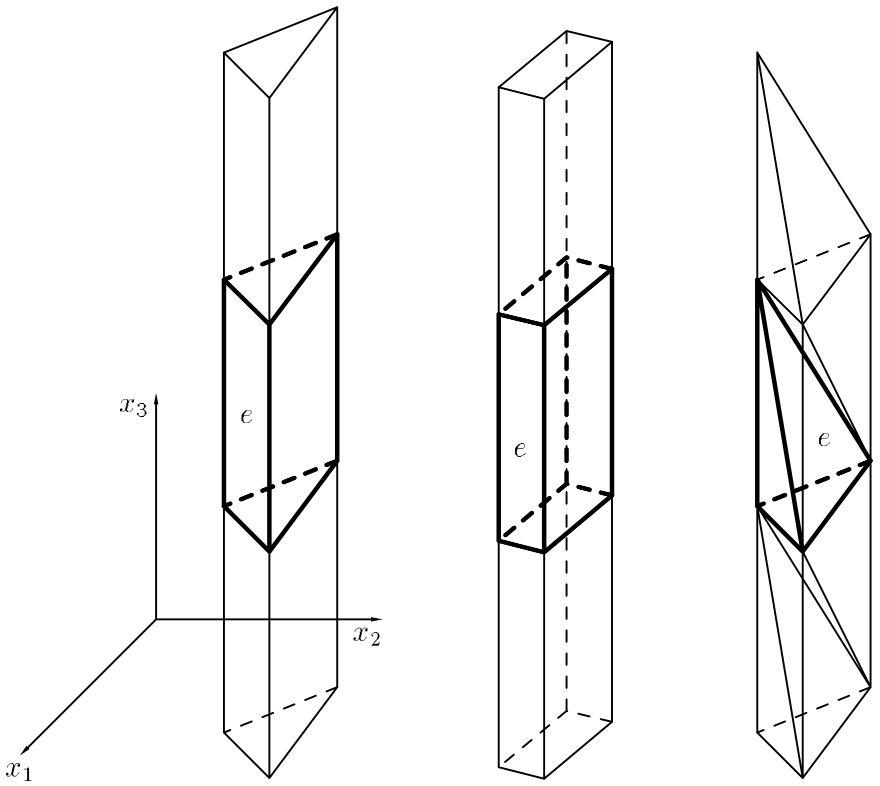


Figure 9: Illustration of the possible choice of a smaller S_e in the case of E_h (three element types).

5.2 Stability in weighted Sobolev spaces

As in Subsection 3.2 we do not have an estimate of type $(1, 1)$ for E_h. Therefore we consider a stability estimate for functions from weighted Sobolev spaces $V_\beta^{\ell,p}(S_e)$. These spaces were introduced in (3.8), (3.9). To be able to apply the transformation (5.2) to the weight we will restrict the consideration to the case $h_1 \sim h_2$. However, we can then relax (5.2) to (1.10), see Remark 1.

Lemma 11 *Let m be an integer and β, p, q be real numbers with $0 \le m \le k$, $p, q \in [1, \infty]$, $\beta < 2 - \frac{2}{p}$, $\beta \le 1$. Then for $u \in W^{m,p}(S_e) \cap V_\beta^{m+1,p}(S_e)$ the stability estimate*

$$|\mathrm{E}_h u; W^{m,q}(e)| \le (\operatorname{meas} e)^{1/q-1/p}\, h_1^{-\beta} \sum_{|\alpha|=m-1} \sum_{|t|=1} h^t \| D^{\alpha+t} v; V_\beta^{1,p}(S_e)\| \tag{5.11}$$

holds if $m \ge 1$ or $p \ge 2$.

Proof Observe that the relations

$$\begin{aligned} \|v; L^1(S_e)\| &\le \|r^{-\beta}; L^{p'}(S_e)\| \|r^\beta v; L^p(S_e)\|, && (5.12)\\ \|r^{-\beta}; L^{p'}(S_e)\| &\lesssim (\operatorname{meas} S_e)^{1-1/p} h_1^{-\beta} && (5.13) \end{aligned}$$

(compare (3.14), (3.15)) lead to the embedding

$$V_\beta^{m+1,p}(S_e) \hookrightarrow V_0^{m+1,1}(S_e) \hookrightarrow W^{m+1,1}(S_e), \quad \beta < 2 - \frac{2}{p},$$

that means $u \in W^{m+1,1}(S_e)$. Therefore we can apply Theorem 10 (see also Remark 1) with $p = 1$:

$$|\mathrm{E}_h u; W^{m,q}(e)| \lesssim (\operatorname{meas} e)^{1/q-1} \sum_{|\alpha|\le 1} h^\alpha |D^\alpha u; W^{m,1}(S_e)| \tag{5.14}$$

Notice further that (5.12), (5.13) lead to the estimate

$$\|v; L^1(S_e)\| \lesssim (\operatorname{meas} S_e)^{1-1/p} h_1^{-\beta} \|r^\beta v; L^p(S_e)\|, \quad \beta < 2 - \frac{2}{p}.$$

So we get

$$\begin{aligned}
&\sum_{|\alpha|\le 1} \sum_{|t|=1} h^\alpha \|D^{\alpha+t} v; L^1(S_e)\| \\
&\lesssim (\operatorname{meas} S_e)^{1-1/p} h_1^{-\beta} \left(\sum_{|\alpha|=1} \sum_{|t|=1} h^\alpha \|r^\beta D^{\alpha+t} v; L^p(S_e)\| + \sum_{|t|=1} h_1 \|r^{\beta-1} D^t v; L^p(S_e)\| \right) \\
&\lesssim (\operatorname{meas} S_e)^{1-1/p} h_1^{-\beta} \sum_{|s|=1} h^s \|D^s v; V^{1,p}(S_e)\|.
\end{aligned}$$

Together with (5.14) the assertion (5.11) is concluded. □

6 Application to the Poisson problem in a domain with an edge

Consider the Poisson problem with in general mixed boundary conditions in a three-dimensional polyhedral domain Ω. It is well known that the solution has in general singularities near corners and edges and near the lines where the type of the boundary condition changes. As a result, the finite element method on quasi-uniform meshes loses accuracy. The rate of convergence is smaller in comparison with that for problems with smooth solutions. To compensate this, specially adapted numerical methods have been developed. The *singular function method* which is well developed for two-dimensional problems is used for three-dimensional problems in [11, 20]. However, *mesh refinement* techniques seem to be easier to handle. Refined *isotropic* meshes were considered in [3, 9, 21] for the finite element method and the boundary element method but this approach leads to overrefinement near edges. This overrefinement can be avoided by using *anisotropic meshes* in the neighbourhood of the edges [2, 8, 24].

In [2, 8] we considered the Dirichlet problem for the Poisson equation over a prismatic domain

$$\Omega = G \times I \tag{6.1}$$

where $G \subset \mathbb{R}^2$ is a bounded polygonal domain and $I := (0, z_0) \subset \mathbb{R}$ is an interval. This restriction was made there because we wanted to focus on *edge singularities*, and such domains do not introduce additional corner singularities [27]. The finite element meshes in [2, 8] were of tensor product type, graded perpendicularly to the edge and quasi-uniform in the edge direction. Pentahedral meshes seem to be natural but in that papers the pentahedra were divided into three tetrahedra each. Pentahedral elements were used in [7], an unpublished version of the paper [8]. Note that this class of domains and the meshes exactly match the assumptions made in Section 1 for the present paper.

The estimation of the finite element error in the energy norm can be reduced to a general approximation problem due to the projection property of the finite element method. In the previous papers the interpolation error was investigated and it was shown that the family of meshes considered there is suited for the treatment of edge singularities. However, two points are still insufficient: First, the assumptions on the regularity of the right hand side

f of the Poisson equation were quite high in [2]. This drawback was partially removed in [8], but the case $f \in L^2(\Omega)$ is still not treated. This is deficient because Nitsche's method for obtaining an $L^2(\Omega)$-estimate of the finite element error is not applicable. Second, the refinement condition in [8] is slightly stronger than in [2]; this seems to be unnecessary. The aim of this section is to prove optimal estimates of the finite element error in the $W^{1,2}(\Omega)$- and the $L^2(\Omega)$-norm for $f \in L^2(\Omega)$ and the weaker refinement condition of [2]. This is now possible due to the local anisotropic estimates for the quasi-interpolation operators.

The plan of this section is the following. First we pose two model problems which differ in their boundary conditions. Then we introduce the family of finite element meshes. The global quasi-interpolation error is estimated in the $W^{1,2}(\Omega)$-seminorm. Because in general the operators do not preserve Dirichlet boundary conditions the model problems are chosen such that in one case S_h and in the other case E_h are appropriate and no modification of the operator is necessary near the boundary. The main result of this section can then be concluded, namely the finite element error estimates. Some remarks on other than the model problems complete this section.

Consider a prismatic domain Ω as described in (6.1) and denote $\Gamma_\mathrm{B} := \{x \in \partial\Omega : x_3 = 0 \text{ or } x_3 = z_0\}$ and $\Gamma_\mathrm{M} := \{x \in \partial\Omega : 0 < x_3 < z_0\} = \partial\Omega \setminus \Gamma_\mathrm{B}$. Then we treat the mixed boundary value problems

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma_\mathrm{B}, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_\mathrm{M}, \tag{6.2}$$

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma_\mathrm{M}, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_\mathrm{B}, \tag{6.3}$$

with $f \in L^2(\Omega)$. We assume that the cross-section G has only one corner with interior angle $\omega > \pi$ at the origin; thus Ω has only one "singular edge" which is part of the x_3-axis. The case of more than one singular edge introduces no additional difficulties because the edge singularities are of local nature.

Let $V_0 \subset W^{1,2}(\Omega)$ be the space of all $W^{1,2}(\Omega)$-functions which vanish at the Dirichlet part of the boundary (different for problems (6.2) and (6.3)), and introduce the bilinear form $a(.,.) : V_0 \times V_0 \to I\!R$ and the linear form $(f,.) : V_0 \to I\!R$ by

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v, \quad (f,v) := \int_\Omega fv$$

The variational form of problems (6.2) and (6.3) is given by

$$\textit{Find } u \in V_0 \textit{ such that } a(u,v) = (f,v) \textit{ for all } v \in V_0. \tag{6.4}$$

The existence of a unique variational solution u follows from the Lax-Milgram lemma.

The properties of the solution u can be described favourably using the weighted Sobolev spaces $V_\beta^{\ell,p}$ introduced in Subsection 3.2.

Lemma 12 *The solutions u of both problems (6.2) and (6.3) satisfy*

$$\frac{\partial u}{\partial x_i} \in V_\beta^{1,2}(\Omega), \qquad \left\| \frac{\partial u}{\partial x_i}; V_\beta^{1,2}(\Omega) \right\| \lesssim \|f; L^2(\Omega)\|, \quad i = 1,2, \ \beta > 1 - \frac{\pi}{\omega}, \tag{6.5}$$

$$\frac{\partial u}{\partial x_3} \in V_0^{1,2}(\Omega), \qquad \left\| \frac{\partial u}{\partial x_3}; V_0^{1,2}(\Omega) \right\| \lesssim \|f; L^2(\Omega)\|. \tag{6.6}$$

Proof The edge singularities can be described by (6.5), (6.6), see for example [18, §26 and §30] or [8, Section 2]. Corner singularities are not present which can be shown by mirror techniques, compare also [27]. □

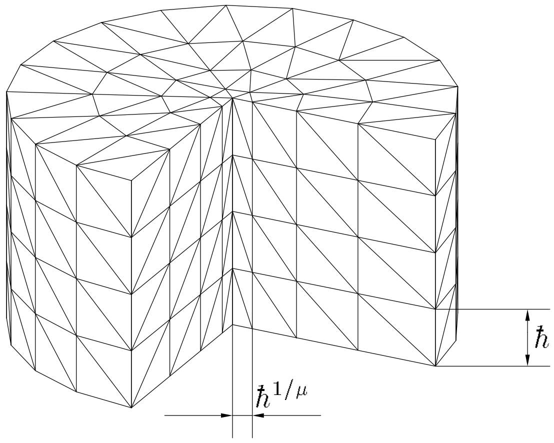


Figure 10: Example for an anisotropic mesh.

We define now a family of meshes $\mathcal{T}_h = \{e\}$ of tensor product type by introducing in G the standard mesh grading for two-dimensional corner problems, see for example [22]. Let $\{\eta\}$ be a regular isotropic triangulation of G; the elements are triangles. With $\hbar$ being the global mesh parameter, $\mu \in (0,1]$ being the grading parameter, r_η being the distance of η to the corner,

$$r_\eta := \min_{(x_1,x_2)\in\overline{\eta}} (x_1^2 + x_2^2)^{1/2},$$

and some constant $R > 0$, we assume that the element size $h_\eta := \operatorname{diam} \eta$ satisfies

$$h_\eta \sim \begin{cases} \hbar^{1/\mu} & \text{for } r_\eta = 0, \\ \hbar r_\eta^{1-\mu} & \text{for } 0 < r_\eta \leq R, \\ \hbar & \text{for } r_\eta > R. \end{cases}$$

This graded two-dimensional mesh is now extended in the third dimension using a uniform mesh size $\hbar$. In this way we obtain a pentahedral or, by dividing each pentahedron, a tetrahedral triangulation of Ω, see Figure 10 for an illustration. Note that the number of elements is of the order $\hbar^{-3}$ for the full range of μ. The notation is extended to the three-dimensional case as follows. Let r_e be the distance of an element e to the edge (x_3-axis). Then the element sizes satisfy

$$h_{1,e} \sim h_{2,e} \sim \begin{cases} \hbar^{1/\mu} & \text{for } r_e = 0, \\ \hbar r_e^{1-\mu} & \text{for } 0 < r_e \leq R, \\ \hbar & \text{for } r_e > R. \end{cases} \qquad h_{3,e} \sim \hbar. \tag{6.7}$$

We introduce now the finite element space $V_{0h} := V_h \cap V_0$ where V_h is defined in Section 1. The finite element solution u_h is determined by

$$\textit{Find } u_h \in V_{0h} \textit{ such that } a(u_h, v_h) = (f, v_h) \textit{ for all } v_h \in V_{0h}. \tag{6.8}$$

Remember that V_{0h} is adapted to the Dirichlet boundary condition and therefore different for Problems (6.2) and (6.3).

Theorem 13 *Let u be the solution of* (6.2). *Then the estimate*

$$|u - \mathrm{S}_h u; W^{1,2}(\Omega)| \lesssim \hbar \, \|f; L^2(\Omega)\|$$

holds if $\mu < \frac{\pi}{\omega}$.

Proof We reduce the estimation of the global error to the evaluation of the local errors and distinguish between the elements far from the edge M and the elements close to M.

For all elements e with $\overline{S_e} \cap M = \emptyset$ we can use Theorem 6 with $m = k = 1$ and $\ell = p = q = 2$:

$$\begin{aligned} |u - \mathrm{S}_h u; W^{1,2}(e)| &\lesssim \sum_{|\alpha|=1} h^\alpha |D^\alpha u; W^{1,2}(S_e)| \\ &\lesssim \sum_{i=1}^{2} h_{i,e} r_e^{-\beta} \left| \frac{\partial u}{\partial x_i}; V_\beta^{1,2}(S_e) \right| + h_{3,e} \left| \frac{\partial u}{\partial x_3}; V_0^{1,2}(S_e) \right| \end{aligned} \tag{6.9}$$

for any $\beta > 1 - \frac{\pi}{\omega}$. Here, we have used the fact that $r_e \lesssim \operatorname{dist}(S_e, M)$ holds, which follows from

$$r_e \le \operatorname{dist}(S_e, M) + h_{1,e'} \sim \operatorname{dist}(S_e, M) + \hbar\, [\operatorname{dist}(S_e, M)]^{1-\mu}$$

for sufficiently small $\hbar$, compare also Figure 3 for an illustration. We apply now the assumption (6.7) and obtain for $r_e \le R$ and $\beta = 1 - \mu$ the relation $h_{i,e} r_e^{-\beta} \sim \hbar r_e^{1-\mu-\beta} = \hbar$ $(i = 1, 2)$. The choice $\beta = 1 - \mu$ is admissible due to the refinement condition $\mu < \frac{\pi}{\omega}$. — In the case $r_e > R$ we have $h_{i,e} r_e^{-\beta} \lesssim \hbar R^{-\beta} \sim \hbar$. Combining this with (6.9) we obtain

$$|u - \mathrm{S}_h u; W^{1,2}(e)| \lesssim \hbar \sum_{i=1}^{2} \left| \frac{\partial u}{\partial x_i}; V_\beta^{1,2}(S_e) \right| + \hbar \left| \frac{\partial u}{\partial x_3}; V_0^{1,2}(S_e) \right|. \tag{6.10}$$

Consider now the elements e with $\overline{S_e} \cap M \neq \emptyset$. We use the triangle inequality and Lemma 7 with $m = k = 1$, $p = 2$, $\beta \in (1 - \frac{\pi}{\omega}, 1)$:

$$\begin{aligned} |u - \mathrm{S}_h u; W^{1,2}(e)| &\lesssim |u; W^{1,2}(e)| + |\mathrm{S}_h u; W^{1,2}(e)| \\ &\lesssim \sum_{|\alpha|=1} \|D^\alpha u, L^2(e)\| + h_{1,e}^{-\beta} \sum_{|\alpha|=1} h^\alpha \|D^\alpha u, V_\beta^{1,2}(S_e)\|. \end{aligned} \tag{6.11}$$

For the first term we use that $r \lesssim h_{1,e}$ in e and $1 - \beta > 0$ and obtain

$$\begin{aligned} \sum_{|\alpha|=1} \|D^\alpha u, L^2(e)\| &\lesssim \sum_{i=1}^{2} h_{1,e}^{1-\beta} \left\| \frac{\partial u}{\partial x_i}; V_{\beta-1}^{0,2}(e) \right\| + h_{1,e} \left\| \frac{\partial u}{\partial x_3}; V_{-1}^{0,2}(e) \right\| \\ &\lesssim \hbar \sum_{i=1}^{2} \left\| \frac{\partial u}{\partial x_i}; V_\beta^{1,2}(e) \right\| + \hbar \left\| \frac{\partial u}{\partial x_3}; V_0^{1,2}(e) \right\|. \end{aligned} \tag{6.12}$$

We also used that $h_{1,e}^{1-\beta} \sim \hbar^{(1-\beta)/\mu} = \hbar$ for $\beta = 1 - \mu$. The second term is treated with similar arguments:

$$\begin{aligned} h_{1,e}^{-\beta} \sum_{|\alpha|=1} h^\alpha \|D^\alpha u, V_\beta^{1,2}(S_e)\| &\lesssim \sum_{i=1}^{2} h_{1,e}^{1-\beta} \left\| \frac{\partial u}{\partial x_i}; V_\beta^{1,2}(S_e) \right\| + h_{1,e}^{-\beta} \hbar \left\| \frac{\partial u}{\partial x_3}; V_\beta^{1,2}(S_e) \right\| \\ &\lesssim \hbar \sum_{i=1}^{2} \left\| \frac{\partial u}{\partial x_i}; V_\beta^{1,2}(S_e) \right\| + \hbar \left\| \frac{\partial u}{\partial x_3}; V_0^{1,2}(S_e) \right\|. \end{aligned} \tag{6.13}$$

The last term was estimated using $r^\beta \le h_{1,e}^\beta$.

Inserting (6.12) and (6.13) in (6.11) we find that (6.10) (with full norms instead of seminorms at the right hand side) holds for elements with $\overline{S_e} \cap M \neq \emptyset$ as well. Summing up over all elements we obtain

$$|u - \mathrm{S}_h u; W^{1,2}(\Omega)| \lesssim \hbar \sum_{i=1}^{2} \left\| \frac{\partial u}{\partial x_i}; V_\beta^{1,2}(\Omega) \right\| + \hbar \left\| \frac{\partial u}{\partial x_3}; V_0^{1,2}(\Omega) \right\|,$$

$\beta = 1-\mu \in (1-\frac{\pi}{\omega}, 1)$. Here we used that only a finite number (independent of $\hbar$) of patches S_e overlap. By applying Lemma 12 the theorem is proved. □

Theorem 14 *Let u be the solution of* (6.3). *Then the estimate*

$$|u - \mathrm{E}_h u; W^{1,2}(\Omega)| \lesssim \hbar\, \|f; L^2(\Omega)\|$$

holds if $\mu < \frac{\pi}{\omega}$.

Proof The theorem can be proved in the same way as Theorem 6.2. Note that we used only the following properties of S_h:

$$\begin{aligned} |u - \mathrm{S}_h u; W^{1,2}(e)| &\lesssim \sum_{|\alpha|=1} h^\alpha |D^\alpha u; W^{1,2}(S_e)|, \\ |\mathrm{S}_h u; W^{1,2}(e)| &\lesssim h_{1,e}^{-\beta} \sum_{|\alpha|=1} \|D^\alpha u, V_\beta^{1,2}(S_e)\|. \end{aligned}$$

Both estimates hold true for E_h as well, see Theorem 10 and Lemma 11. □

Corollary 15 *Let u be the solution of (6.2) or (6.3) and let u_h be the finite element solution defined by (6.8). Assume that the mesh is refined according to $\mu < \frac{\pi}{\omega}$. Then the finite element error can be estimated by*

$$\begin{aligned} |u - u_h; W^{1,2}(\Omega)| &\lesssim \hbar\, \|f; L^2(\Omega)\|, \\ \|u - u_h; L^2(\Omega)\| &\lesssim \hbar^2\, \|f; L^2(\Omega)\|. \end{aligned}$$

Proof The first estimate follows from Theorems 13 and 14 via the projection property of the finite element method. Note that $\mathrm{S}_h u \in V_{0h}$ in the case of problem (6.2) and $\mathrm{E}_h u \in V_{0h}$ for (6.3). The $L^2(\Omega)$-estimate is obtained by Nitsche's method. □

By analogy one can prove for $\frac{\pi}{\omega} < \mu \le 1$ that

$$\begin{aligned} |u - u_h; W^{1,2}(\Omega)| &\lesssim \hbar^{\pi/(\mu\omega)-\varepsilon}\, \|f; L^2(\Omega)\|, \\ \|u - u_h; L^2(\Omega)\| &\lesssim \hbar^{2[\pi/(\mu\omega)-\varepsilon]}\, \|f; L^2(\Omega)\|, \end{aligned}$$

for arbitrary small $\varepsilon > 0$. That means that we get for the unrefined mesh ($\mu = 1$) only an approximation order $\frac{\pi}{\omega} - \varepsilon$ ($W^{1,2}(\Omega)$-norm) or $2(\frac{\pi}{\omega} - \varepsilon)$ ($L^2(\Omega)$-norm). We conjecture that the ε can be omitted. But this needs another way of proof, for example using the theory of interpolation spaces, compare [10] for the two-dimensional case. However, one can show by an example that these estimates cannot be improved further [1]. Numerical tests support the results, see [2, 6, 7].

In the same way as above on can treat certain other boundary conditions. Conditions of third kind impose no further difficulties. Moreover, we can treat cases where Dirichlet boundary conditions are given only on a part of either Γ_{B} or Γ_{M}. In particular, if the type of the boundary condition changes at the edge M we have to substitute the expression $\frac{\pi}{\omega}$ by $\frac{\pi}{2\omega}$ in the whole text. Note further that for $\omega \ge \pi$ the solution is not any more contained in $W^{3/2+\varepsilon,2}(\Omega)$ which implies that the interpolation operator I_h is not applicable to u.

However, if Dirichlet boundary conditions are given on (parts of) both Γ_{B} and Γ_{M} then neither $\mathrm{S}_h u \in V_{0h}$ nor $\mathrm{E}_h u \in V_{0h}$. In such cases we have to modify S_h or E_h near the Dirichlet boundary, as it was done by Clement for C_h [16]. But we will not develop this here.

<table>
<tr><th></th><th>Z_h</th><th>S_h</th><th>L_h</th><th>E_h</th></tr>
<tr><td>2D</td><td>tensor product
h_1, h_2 arbitrary</td><td>tensor product
$h_1 \lesssim h_2$</td><td>tensor product
$h_1 \gtrsim h_2$</td><td></td></tr>
<tr><td rowspan="2">3D</td><td>tensor prod. type
$h_1 \sim h_2 \lesssim h_3$
or $h_1 \sim h_2 \gtrsim h_3$</td><td rowspan="2">tensor prod. type
$h_1 \sim h_2 \lesssim h_3$</td><td rowspan="2">tensor prod. type
$h_1 \sim h_2 \gtrsim h_3$</td><td>tensor prod. type
$h_1 \sim h_2 \lesssim h_3$</td></tr>
<tr><td>tensor product
h_1, h_2, h_3 independent</td><td>tensor product
$h_1 \lesssim h_2 \lesssim h_3$</td></tr>
</table>

Table 1: Tractable finite elements.

<table>
<tr><th>Z_h</th><th>S_h</th><th>L_h</th><th>E_h</th></tr>
<tr><td rowspan="3">$m = 0$
$1 \leq \ell \leq k+1$
$p, q \in [1, \infty]$</td><td rowspan="3">$0 \leq m \leq \ell - 1$
$1 \leq \ell \leq k+1$
$p, q \in [1, \infty]$
for $m \geq 2$ triangles and tetrahedra are excluded</td><td rowspan="3">$0 \leq m \leq \ell$
$1 \leq \ell \leq k+1$
$p, q \in [1, \infty]$</td><td>$1 \leq m \leq \ell - 1$
$1 \leq \ell \leq k+1$
$p, q \in [1, \infty]$</td></tr>
<tr><td>$m = 0$
$2 \leq \ell \leq k+1$
$p, q \in [1, \infty]$</td></tr>
<tr><td>$m = 0$, $\ell = 1$
$p \in (2, \infty]$
$q \in [1, \infty]$</td></tr>
</table>

Table 2: Conditions for the stability and error estimates.

7 Summary

The starting point of our investigation was the quasi-interpolation operator Z_h introduced by Scott and Zhang [25]. We have seen in Section 2 that anisotropic estimates of type (m, ℓ) are valid for $m = 0$ but in general not for $m \geq 1$. Therefore we introduced three modifications and investigated the resulting operators S_h, L_h, and E_h, for the definitions see pages 10, 17, and 22. To summarize and to compare the different Scott-Zhang type quasi-interpolation operators we give a tabular overview. In Table 1 we find the element types which the operator is applicable for. Note the slight difference of *tensor product type* and *tensor product* elements in three dimensions. Tensor product type corresponds to transformation (1.10), (1.11), and tensor product means the restriction to transformation (5.2).

Table 2 compares the conditions for which the stability estimate

$$|\mathrm{Q}_h u; W^{m,q}(e)| \lesssim (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha| \leq \ell - m} h^\alpha |D^\alpha u; W^{m,p}(S_e)|$$

holds, $\mathrm{Q}_h \in \{\mathrm{Z}_h, \mathrm{S}_h, \mathrm{L}_h, \mathrm{E}_h\}$. In the case of S_h and E_h we additionally proved stability in weighted Sobolev spaces. The estimate

$$|\mathrm{Q}_h u; W^{m,q}(e)| \leq (\operatorname{meas} e)^{1/q-1/p} h_1^{-\beta} \sum_{|\alpha| = m-1} \sum_{|t|=1} h^t \| D^{\alpha+t} u; V_\beta^{1,p}(S_e) \|$$

holds under the conditions given in Table 3.

Z_h	S_h	L_h	E_h
not treated	$0 \le m \le k$ $p, q \in [1, \infty]$ $\beta < 2 - \frac{2}{p}$, $\beta \le 1$ for $m \ge 2$ triangles and tetrahedra are excluded	not treated	$1 \le m \le k$ $p, q \in [1, \infty]$ $\beta < 2 - \frac{2}{p}$, $\beta \le 1$
			$m = 0$ $p \in (2, \infty]$ $q \in [1, \infty]$ $\beta < 2 - \frac{2}{p}$, $\beta \le 1$

Table 3: Conditions for the stability in weighted Sobolev spaces.

Z_h	S_h	L_h	E_h
only $m = 0$	$m = \ell$ excluded		$m = \ell$ excluded
	only $m = 0, 1$ for simplices		restrictions on ℓ, p, q for $m = 0$
	in 3D only needle elements	in 3D only flat elements	

Table 4: Restrictions in the applicability of the operators.

The approximation error estimate

$$|u - \mathrm{Q}_h u; W^{m,q}(e)| \lesssim (\operatorname{meas} e)^{1/q-1/p} \sum_{|\alpha|=\ell-m} h^\alpha |D^\alpha u; W^{m,p}(S_e)|$$

holds if the conditions of Table 2 are satisfied and the parameters ℓ, p, m, q are such that the embedding $W^{\ell,p}(e) \hookrightarrow W^{m,q}(e)$ holds.

Some shortcomings of the operators are given in Table 4. Additionally, we state that Dirichlet boundary conditions $u = g \in V_h|_{\Gamma_1}$ on Γ_1 can be satisfied on any part of $\partial\Omega$ for Z_h, on parts of the boundary which are parallel to the x_1-axis/x_1, x_2-plane for S_h and L_h, and on parts of $\partial\Omega$ which are perpendicular to the x_1, x_2-plane for E_h.

Finally, we mention that S_h and E_h have been successfully applied in the study of the Poisson problem in a domain with an edge where the singularity was treated with anisotropic mesh refinement, see Section 6. The operator L_h was applied by Becker [12] to show the stability and an approximation error estimate of the stabilized Q_1/Q_0-element pair in the context of the Stokes equation.

Acknowledgement. The work of the author is supported by DFG (German Research Foundation), Sonderforschungsbereich 393.

References

[1] Th. Apel. *Finite-Elemente-Methoden über lokal verfeinerten Netzen für elliptische Probleme in Gebieten mit Kanten.* PhD thesis, TU Chemnitz, 1991.

[2] Th. Apel and M. Dobrowolski. Anisotropic interpolation with applications to the finite element method. *Computing*, 47:277–293, 1992.

[3] Th. Apel and B. Heinrich. Mesh refinement and windowing near edges for some elliptic problem. *SIAM J. Numer. Anal.*, 31:695–708, 1994.

[4] Th. Apel and G. Lube. Anisotropic mesh refinement for singularly perturbed reaction diffusion problems. Preprint SFB393/96-11, TU Chemnitz-Zwickau, 1996.

[5] Th. Apel and G. Lube. Anisotropic mesh refinement in stabilized Galerkin methods. *Numer. Math.*, 74:261–282, 1996.

[6] Th. Apel and F. Milde. Comparison of several mesh refinement strategies near edges. *Comm. Numer. Meth. Engrg.*, 12:373–381, 1996.

[7] Th. Apel and S. Nicaise. The finite element method with anisotropic mesh grading for the poisson problem in domains with edges. Technical report, TU Chemnitz-Zwickau, 1994. Available only via ftp, server ftp.tu-chemnitz.de, directory pub/Local/mathematik/Apel, file an1.ps.Z.

[8] Th. Apel and S. Nicaise. Elliptic problems in domains with edges: anisotropic regularity and anisotropic finite element meshes. In J. Cea, D. Chenais, G. Geymonat, and J. L. Lions, editors, *Partial Differential Equations and Functional Analysis (In Memory of Pierre Grisvard)*, pages 18–34. Birkhäuser, Boston, 1996.

[9] Th. Apel, A.-M. Sändig, and J. R. Whiteman. Graded mesh refinement and error estimates for finite element solutions of elliptic boundary value problems in non-smooth domains. *Math. Meth. Appl. Sci.*, 19:63–85, 1996.

[10] I. Babuška, R. B. Kellogg, and J. Pitkäranta. Direct and inverse error estimates for finite elements with mesh refinements. *Numer. Math.*, 33:447–471, 1979.

[11] A. E. Beagles and J. R. Whiteman. Finite element treatment of boundary singularities by augmentation with non-exact singular functions. *Numer. Methods Partial Differ. Equations*, 2:113–121, 1986.

[12] R. Becker. *An adaptive finite element method for the incompressible Navier–Stokes equations on time-dependent domains.* PhD thesis, Ruprecht-Karls-Universität Heidelberg, 1995.

[13] J. H. Bramble and S. R. Hilbert. Estimation of linear functionals on Sobolev spaces with applications to Fourier transforms and spline interpolation. *SIAM J. Numer. Anal.*, 7:112–124, 1970.

[14] J. H. Bramble and S. R. Hilbert. Bounds for a class of linear functionals with applications to Hermite interpolation. *Numerische Mathematik*, 16:362–369, 1971.

[15] P. Ciarlet. *The finite element method for elliptic problems.* North-Holland, Amsterdam, 1978.

[16] P. Clement. Approximation by finite element functions using local regularization. *RAIRO Anal. Numer.*, 2:77–84, 1975.

[17] T. Dupont and R. Scott. Polynomial approximation of functions in Sobolev spaces. *Math. Comp.*, 34:441–463, 1980.

[18] A. Kufner and A.-M. Sändig. *Some Applications of Weighted Sobolev Spaces.* BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1987.

[19] G. Kunert. Ein Residuenfehlerschätzer für anisotrope Tetraedernetze und Dreiecksnetze in der Finite-Elemente-Methode. Preprint SPC 95_10, TU Chemnitz–Zwickau, 1995.

[20] M. S. Lubuma and S. Nicaise. Finite element method for elliptic problems with edge corners. *J. Math. Anal. Appl.* Submitted.

[21] M. S. Lubuma and S. Nicaise. Dirichlet problems in polyhedral domains II: approximation by FEM and BEM. *J. Comp. Appl. Math.*, 61:13–27, 1995.

[22] L. A. Oganesyan and L. A. Rukhovets. *Variational-difference methods for the solution of elliptic equations.* Izd. Akad. Nauk Armyanskoi SSR, Jerevan, 1979. (Russian).

[23] P. Oswald. *Multilevel Finite Element Approximation: Theory and Applications.* Teubner, Stuttgart, 1994.

[24] T. von Petersdorff. *Randwertprobleme der Elastizitätstheorie für Polyeder — Singularitäten und Approximationen mit Randelementmethoden.* PhD thesis, TH Darmstadt, 1989.

[25] L. R. Scott and S. Zhang. Finite element interpolation of non-smooth functions satisfying boundary conditions. *Math. Comp.*, 54:483–493, 1990.

[26] K. Siebert. An a posteriori error estimator for anisotropic refinement. *Numer. Math.*, 73:373–398, 1996.

[27] E. Stephan and J. R. Whiteman. Singularities of the Laplacian at corners and edges of three–dimensional domains and their treatment with finite element methods. *Math. Meth. Appl. Sci.*, 10(3):339–350, 1988.

The complete list of the preprints of Sonderforschungsbereich 393 is available via `http://www.tu-chemnitz.de/sfb393/sfb97pr.html`.