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Numerische Simulation auf massiv parallelen Rechnern

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Anisotropic mesh refinement for singularly perturbed reaction diffusion problems

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Abstract. The paper is concerned with the finite element resolution of layers appearing in singularly perturbed problems. A special anisotropic grid of Shishkin type is constructed for reaction diffusion problems. Estimates of the finite element error in the energy norm are derived for two methods, namely the standard Galerkin method and a stabilized Galerkin method. The estimates are uniformly valid with respect to the (small) diffusion parameter. One ingredient is a pointwise description of derivatives of the continuous solution. A numerical example supports the result.

Another key ingredient for the error analysis is a refined estimate for (higher) derivatives of the interpolation error. The assumptions on admissible anisotropic finite elements are formulated in terms of geometrical conditions for triangles and tetrahedra. The application of these estimates is not restricted to the special problem considered in this paper.

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1 Introduction

In this paper, we consider *reaction diffusion problems* of the form

$$L_\varepsilon u \equiv -\varepsilon^2 \Delta u + cu = f \quad \text{in } \Omega \subset I\!R^d,\ d=2,3, \tag{1.1}$$
$$u = 0 \quad \text{on } \partial\Omega \tag{1.2}$$

where Ω is a bounded polyhedral domain with an at least Lipschitzian boundary $\partial\Omega$ and $\varepsilon \in (0,1]$ is the diffusion parameter. In the *singularly perturbed case* $\varepsilon \ll 1$ the solutions of (1.1), (1.2) are characterized by boundary and/or interior layers of width $\mathcal{O}(\varepsilon \ln \frac{1}{\varepsilon})$; see [13, 15]. This is caused by the fact that the solution u_0 of the algebraic limit equation

$$c(x)u_0(x) = f(x) \quad \text{in } \overline{\Omega}$$

in general cannot satisfy the given boundary condition (1.2) and/or is possibly non-smooth, that means $u_0 \notin W_0^{1,2}(\Omega)$.

For the *standard Galerkin* finite element method and a recent modification of it (hereafter refered to as *Stabilized Galerkin method*), see [12], we try to obtain global discretization error estimates in the energy norm which are *uniformly valid with respect to the full range of* ε. More precisely, a family of approximations u_h, $0 < h \le h_0$, converges uniformly to u in the norm $\|\cdot\|_*$ of order p if

$$\|u - u_h\|_* \le Ch^p$$

with a constant C independent of ε and the discretization parameter h. Furthermore, it is the goal of the present paper to resolve the boundary layer, that means the error must be low there.

We summarize now some previous results on the numerical solution of singularly perturbed reaction diffusion problems in multiple dimensions. Galerkin type finite element methods were analyzed mainly for *isotropic* meshes, that means $h_{1,e}/\varrho_e = \mathcal{O}(1)$ for $\varepsilon \to 0$, $h \to 0$, where $h_{1,e}$ and ϱ_e denote the diameter of the finite element e and the diameter of the largest inscribed ball in e, respectively. Schatz and Wahlbin [20] analyzed carefully two (and one) dimensional problems. The key results are global and local L^∞-error estimates. Furthermore, under some regularity assumptions to the data L^2-estimates are derived which are uniformly valid in ε. Also the case of rough data is addressed. — Stabilized variants of the Galerkin method with additional weighted residual terms have been considered for example by Franca and coworkers in [12]. Whereas the analysis of schemes with piecewise linear elements is in some sense clear on isotropic meshes [12], this is seemingly not the case for higher order elements; we will discuss this in Section 4 of the present paper. — In our former paper [3] we studied the Galerkin/Least-squares type stabilization for a convection diffusion reaction model. This approach is not suited to the case of a pure reaction diffusion problem. The optimization of the numerical diffusion parameters δ_e leads to $\delta_e = 0$ for all e, that is a pure Galerkin method.

We remark that no attempt is made to resolve the layers in the above mentioned papers [12, 20]. A resolution of boundary and interior layers with isotropic elements leads to overrefinement. *Anisotropic* mesh refinement in the sense $\lim_{\varepsilon\to+0} h_{1,e}/\varrho_e = \infty$ is much more efficient in such thin layers. Previous results concerning the resolution of boundary layers for the problem under consideration are due to Shishkin [21, 22] in the context of finite difference methods in two and three dimensions, due to Blatov [9] in the context of the h-version of the finite element method (bilinear elements), and due to Xenophontos [25] for the hp-version of the finite element method, both in two dimensions only. In [9, 21] the authors used meshes of *Bakhvalov type* [6], and in [22] similar but simpler *Shishkin type* meshes. Both types of meshes are isotropic away from layers and anisotropically refined close to the manifold where the layer is located. The error estimates were derived in the maximum norm [9, 21, 22], see also [19], or in the energy norm [25]. In a recent paper [23], Stynes and O'Riordan derive error estimates in the energy and maximum norm for the diffusion-convection-reaction problem on the unit square using the Galerkin method on a Shishkin mesh with bilinear finite elements.

The key ingredients of the estimates in [9, 21, 22] are pointwise estimates of derivatives of the solution which depend on ε and the distance to the boundary. The theory in [9] uses the Butuzov expansion [10] of the solution, see also [14], whereas Shishkin derives a different representation [22] which suits better for our application, see Subsection 2.2. The analysis in [25] relies on certain tensor product representations of the layer terms in the solution which can be derived in the case $c = \text{const.}$, $f = \text{const.}$ We remark that the case of a smooth domain is simpler concerning the analytic properties of the solution. But it needs other ingredients in the numerical treatment, see [7, 8, 25], for example the representation of the domain in a boundary fitted coordinate system which is hard in general situations.

In this paper, we extend the numerical analysis of Galerkin type finite element methods to meshes which are anisotropically refined at least in boundary layers. In particular, we derive error estimates in the energy norm for the the two- and three-dimensional cases. The ingredients of this analysis are *localized Sobolev norm estimates* of the solution with respect to the diffusion parameter, and sharp *local interpolation error estimates* which reflect and take advantage of the *anisotropic* character of the mesh.

The *outline of the paper* is as follows: In Section 2 we consider problem (1.1), (1.2) for the special case

$$\Omega = (0,1)^d. \tag{1.3}$$

The Galerkin finite element method on anisotropically refined meshes is analyzed for piecewise linear or higher degree shape functions and for different assumptions on the solution u of (1.1), (1.2). A numerical test example is given.

The analysis of anisotropically refined meshes is based on sharp local estimates of the interpolation error in the norm of the Sobolev spaces $W^{1,p}(e)$ as given in [1]. However, for the analysis of the Stabilized Galerkin method we have to extend these results to norms in $W^{2,p}(e)$. Thus we shall treat in Section 3 the case of general Sobolev norms $W^{m,p}(.)$, $m = 0, \ldots, k$. Moreover, in order to deal with more general domains than (1.3) we will introduce geometric conditions to the elements such that the anisotropic interpolation error estimates still hold. In [1], this has been done unsatisfactorily for the three-dimensional case.

Section 4 is devoted to a careful analysis of the Stabilized Galerkin method for problem (1.1), (1.2), (1.3). First we prove existence and uniqueness of the discrete solution. Then we study the finite element error in relation with the choice of the stabilization parameters δ_e. In a short final section we discuss more general domains.

Note that we use the symbol C for a generic positive constant, that means, C may be of different value at each occurrence. But C is always independent of the function under consideration, of the finite element mesh, and particularly of ε. On the contrary, some constants are indexed with a letter for later reference to them.

2 The Galerkin method for the model problem

2.1 The setting of the problem

We consider the singularly perturbed elliptic boundary value problem (1.1), (1.2), with the basic assumptions

(H.1) $0 < \varepsilon \leq 1$, $c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$,

(H.2) $\inf_\Omega c(x) \geq \gamma > 0$.

With $V \equiv W_0^{1,2}(\Omega)$ the *variational formulation* of this problem reads

$$\text{Find } u \in V, \text{ such that } B_G(u,v) = L_G(v) \quad \forall v \in V \tag{2.1}$$

where

$$B_G(u,v) \equiv \varepsilon^2(\nabla u, \nabla v)_\Omega + (cu, v)_\Omega, \tag{2.2}$$
$$L_G(v) \equiv (f, v)_\Omega, \tag{2.3}$$

and $(\cdot,\cdot)_G$ denotes the inner product in $L^2(G)$, $G \subseteq \Omega$.

Let now $\mathcal{T}_h = \{e\}$ be an admissible triangulation of $\overline{\Omega} = \bigcup_e \overline{e}$, that means, let properties $(\mathcal{T}_h 1) - (\mathcal{T}_h 5)$ of [11] be fulfilled. For the moment, we do not need a condition on the angles of the elements. Let $\mathcal{P}_k(e)$ be the space of polynomials of maximal degree $k \geq 1$, defined over e. We introduce the finite element space

$$V_h \equiv \{v \in V : v|_e \in \mathcal{P}_k(e) \;\; \forall e \in \mathcal{T}_h\}. \tag{2.4}$$

Then the *standard Galerkin method* (G) of (2.1) is given by

$$\text{Find } u_h \in V_h, \text{ such that } B_G(u_h, v_h) = L_G(v_h) \quad \forall v_h \in V_h. \tag{G}$$

Introduce the usual norms and seminorms in Sobolev spaces $W^{m,p}(e)$, $m \in I\!N$, $p \in [1,\infty]$, by

$$\|v; W^{m,p}(e)\|^p \equiv \sum_{|\alpha|\leq m} \int_e |D^\alpha v|^p dx, \qquad |v; W^{m,p}(e)|^p \equiv \sum_{|\alpha|=m} \int_e |D^\alpha v|^p dx,$$

with the usual modification for $p = \infty$. Here, we have used a multi-index notation with

$$\alpha = (\alpha_1, \ldots, \alpha_d), \quad |\alpha| = \alpha_1 + \ldots + \alpha_d, \quad D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}},$$

where the numbers α_i $(i = 1, \ldots, d)$ are non-negative integers. Define now the energy norm $|||\cdot|||$ by

$$|||\, v \,|||^2 \equiv B_G(v,v) = \varepsilon^2 |v; W^{1,2}(\Omega)|^2 + \|\sqrt{c}v; L^2(\Omega)\|^2. \tag{2.5}$$

Standard analysis gives the following quasi-optimal global energy norm error estimate.

Theorem 2.1 *Under the assumptions (H.1), (H.2) the estimate*

$$|||\, u - u_h \,||| \leq \inf_{w_h \in V_h} |||\, u - w_h \,||| \tag{2.6}$$

holds for the solution $u \in V$ of (2.1) and for the solution $u_h \in V_h$ of (G) on an arbitrary admissible mesh.

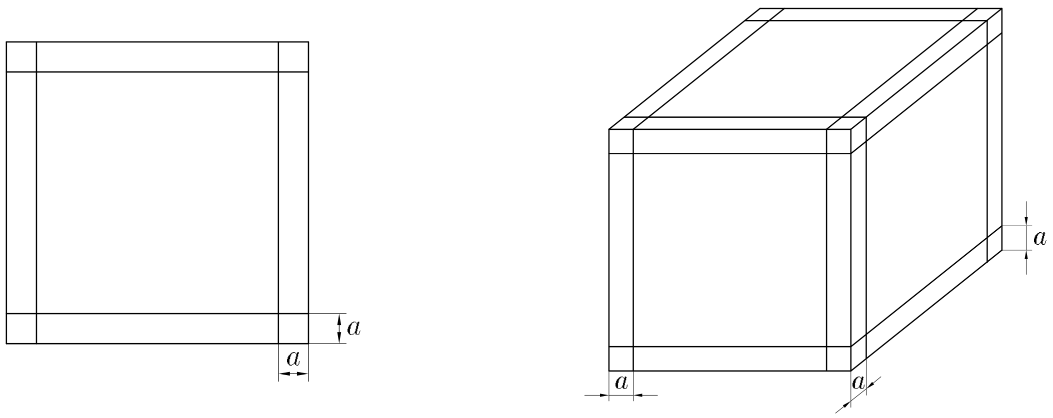

Figure 2.1: Decomposition of the domain $\Omega = (0,1)^d$, $d = 2, 3$.

Using this theorem we can immediately conclude an error estimate for *isotropic quasi-uniform meshes*, namely

$$|||\, u - u_h \,||| \leq C h^k (\varepsilon + h) |u; W^{k+1,2}(\Omega)|.$$

The drawback is that this estimate is not uniform in ε because the seminorm (if it exists) at the right hand side can grow to infinity for $\varepsilon \to 0$.

From now on we will simplify the considerations by restricting to the case (1.3), $\Omega = (0,1)^d$, and the concentration on boundary layers. More general domains are discussed in Section 5. For a special treatment of the boundary layer we introduce a domain decomposition of Ω as illustrated in Figure 2.1. The parameter a will be determined later. Each of the 3^d subdomains is meshed uniformly into n^d rectangles/cubes which are then divided into $d!$ simplices each. In every element, d edges can be chosen which are parallel to the d coordinate axes, respectively. Their lengths are denoted by $h_{1,e}, \ldots, h_{d,e}$, and we define the multi-index notation $h_e^\alpha = h_{1,e}^{\alpha_1} \cdots h_{d,e}^{\alpha_d}$. In this way we get $d+1$ types of elements, namely isotropic elements in the interior and in the corner subdomains as well as anisotropic elements near edges and faces.

Introduce the Lagrangian interpolation operator $I_h^{(k)} : C(\overline{\Omega}) \to V_h$. Then it was proved in [1] that the interpolation error can be estimated by

$$|v - I_h^{(k)} v; W^{m,p}(e)|^p \leq C \sum_{|\alpha| = k+1-m} h_e^{\alpha p} |D^\alpha v; W^{m,p}(e)|^p, \quad m = 0, 1, \tag{2.7}$$

provided that $d = 2$ or $k > m$ or $p > 2$. This estimate holds under geometrical conditions on the element e, see also Subsections 3.2 and 3.4, which are obviously satisfied for our special mesh.

The aim is now to use (2.7) for an estimation of $|||\, u - I_h^{(k)} u \,|||$ in order to get via Theorem 2.1 an error bound of $|||\, u - u_h \,|||$ for an appropriate choice of the parameter a. To obtain an ε-uniform estimate it is necessary to have information on the local behaviour of the exact solution u. We will discuss this in a separate subsection.

2.2 Behaviour of the exact solution

Consider first the two-dimensional case. In order to describe the solution we denote by Γ_ℓ, $\ell = 1, \ldots, 4$, the edges of Ω, that means $\partial\Omega = \bigcup_{\ell=1}^4 \overline{\Gamma_\ell}$. Let $\Gamma_{\ell,m} \equiv \overline{\Gamma_\ell} \cap \overline{\Gamma_m}$, $\ell, m = 1, \ldots, 4$, $m > \ell$, be (if not empty) the corners of Ω. Following [22] one can split the solution u as follows:

$$u = U + \sum_{\ell=1}^{4} V_\ell + \sum_{\substack{\ell,m=1\\ m>\ell}}^{4} V_{\ell,m} \tag{2.8}$$

where U is the "regular part", and V_ℓ, $V_{\ell,m}$ denote the boundary layer parts with respect to the edges Γ_ℓ and the corners $\Gamma_{\ell,m}$.

For the description of the properties of the functions introduce the following boundary fitted Cartesian coordinate systems. Relative to the edges Γ_ℓ, $\ell = 1,\ldots,4$, denote by $x_{1,\ell}, x_{2,\ell}$ a coordinate system with $x_{2,\ell} \equiv \operatorname{dist}(x,\Gamma_\ell)$, for each corner $\Gamma_{\ell,m}$, $\ell,m = 1,\ldots,4$, $m > \ell$, introduce $x_{1,\ell,m}, x_{2,\ell,m}$ with $x_{1,\ell,m} \equiv \operatorname{dist}(x,\Gamma_\ell)$ and $x_{2,\ell,m} \equiv \operatorname{dist}(x,\Gamma_m)$. D^α is to understand with respect to the corresponding coordinate system.

Theorem 2.2 *If the data are sufficiently smooth, $c, f \in C^s(\overline{\Omega})$, $s > 5$, and if certain compatibility conditions on the data are satisfied, then the estimates*

$$\begin{aligned} |D^\alpha U(x)| &\le M\left(1+\varepsilon^{2-|\alpha|}\right) && (2.9)\\ |D^\alpha V_\ell(x)| &\le M\left(\varepsilon^{1-|\alpha|}+\varepsilon^{-\alpha_2}\right)e^{-\gamma_0 x_{2,\ell}/\varepsilon} && (2.10)\\ |D^\alpha V_{\ell,m}(x)| &\le M\varepsilon^{-|\alpha|}\min_{s=1,2} e^{-\gamma_0 x_{s,\ell,m}/\varepsilon} && (2.11)\end{aligned}$$

hold for $|\alpha| \le 3$, $\ell,m = 1,\ldots,4$, $m > \ell$, and for any $\gamma_0 \in (0,\gamma)$; for γ see (H.2).

Proof See [22, pp. 63–68]. □

Corollary 2.3 *Denote $\Omega_1 \equiv (a,1-a)\times(a,1-a)$, $\Omega_2 \equiv (a,1-a)\times(0,a)$, and $\Omega_3 \equiv (0,a)\times(0,a)$, and let the assumptions of Theorem 2.2 be true. Then for $2 \le |\alpha| \le 3$ the estimates*

$$\begin{aligned} \|D^\alpha u; L^2(\Omega_1)\|^2 &\le C\varepsilon^{2(2-|\alpha|)-q} && \text{if } a \ge \tfrac{3-q}{2}\tfrac{\varepsilon}{\gamma_0}\ln\tfrac{1}{\varepsilon}, \quad q \ge 0, && (2.12)\\ \|D^\alpha u; L^2(\Omega_2)\|^2 &\le C\varepsilon^{1-q}\left(\varepsilon^{1-|\alpha|}+\varepsilon^{-\alpha_2}\right)^2 && \text{if } a \ge \tfrac{2-q}{2}\tfrac{\varepsilon}{\gamma_0}\ln\tfrac{1}{\varepsilon}, \quad q \ge 0, && (2.13)\\ \|D^\alpha u; L^2(\Omega_3)\|^2 &\le Ca\varepsilon^{1-2|\alpha|} && \forall a \in \left(0,\tfrac{1}{2}\right) && (2.14)\end{aligned}$$

hold. All other cases of our domain decomposition are equivalent to one of the given cases.

Proof The assertion is proved by integration using the splitting (2.8) and the pointwise estimates of Theorem 2.2. □

The three-dimensional case is considered by analogy. Here, Γ_ℓ, $\ell = 1,\ldots,6$, are the faces of Ω, $\Gamma_{\ell,m} \equiv \overline{\Gamma_\ell}\cap\overline{\Gamma_m}$, $\ell,m = 1,\ldots,6$, $m > \ell$, are (if not empty) the edges, and $\Gamma_{\ell,m,n} \equiv \overline{\Gamma_\ell}\cap\overline{\Gamma_m}\cap\overline{\Gamma_n}$, $\ell,m,n = 1,\ldots,6$, $n > m > \ell$, are (if not empty) the corners of Ω. Relative to the faces Γ_ℓ, $\ell = 1,\ldots,6$, denote by $x_{1,\ell}, x_{2,\ell}, x_{3,\ell}$ a Cartesian coordinate system with $x_{3,\ell} \equiv \operatorname{dist}(x,\Gamma_\ell)$, for each edge $\Gamma_{\ell,m}$, $\ell,m = 1,\ldots,6$, $m > \ell$, introduce $x_{1,\ell,m}, x_{2,\ell,m}, x_{3,\ell,m}$ with $x_{2,\ell,m} \equiv \operatorname{dist}(x,\Gamma_\ell)$ and $x_{3,\ell,m} \equiv \operatorname{dist}(x,\Gamma_m)$, and for each corner $\Gamma_{\ell,m,n}$, $\ell,m,n = 1,\ldots,6$, $n > m > \ell$, introduce $x_{1,\ell,m,n}, x_{2,\ell,m,n}, x_{3,\ell,m,n}$ with $x_{1,\ell,m,n} \equiv \operatorname{dist}(x,\Gamma_\ell)$, $x_{2,\ell,m,n} \equiv \operatorname{dist}(x,\Gamma_m)$, and $x_{3,\ell,m,n} \equiv \operatorname{dist}(x,\Gamma_n)$. Using the representation

$$u = U + \sum_\ell V_\ell + \sum_{\ell,m} V_{\ell,m} + \sum_{\ell,m,n} V_{\ell,m,n} \tag{2.15}$$

we have the following pointwise estimate.

Theorem 2.4 *If the data are sufficiently smooth, $c, f \in C^s(\overline{\Omega})$, $s > 5$, and if certain compatibility conditions on the data are satisfied, then the estimates*

$$\begin{aligned} |D^\alpha U(x)| &\le M\left(1+\varepsilon^{2-|\alpha|}\right) && (2.16)\\ |D^\alpha V_\ell(x)| &\le M\left(\varepsilon^{1-|\alpha|}+\varepsilon^{-\alpha_3}\right)e^{-\gamma_0 x_{3,\ell}/\varepsilon} && (2.17)\\ |D^\alpha V_{\ell,m}(x)| &\le M\left(\varepsilon^{1-|\alpha|}+\varepsilon^{-(\alpha_2+\alpha_3)}\right)\min_{s=2,3} e^{-\gamma_0 x_{s,\ell,m}/\varepsilon} && (2.18)\\ |D^\alpha V_{\ell,m,n}(x)| &\le M\varepsilon^{-|\alpha|}\min_{s=1,2,3} e^{-\gamma_0 x_{s,\ell,m,n}/\varepsilon} && (2.19) \end{aligned}$$

hold for $|\alpha| \le 3$, $\ell, m, n = 1, \ldots, 6$, $n > m > \ell$, and for any $\gamma_0 \in (0, \gamma)$; for γ see (H.2).

Proof See [22, pp. 63–68]. □

The Sobolev norm estimates will be needed in spaces $W^{k+1,p}(\Omega)$, $p \ge 2$.

Corollary 2.5 *By analogy to Corollary 2.3 denote $\Omega_1 \equiv (a, 1-a)^3$, $\Omega_2 \equiv (a, 1-a)^2 \times (0, a)$, $\Omega_3 \equiv (a, 1-a) \times (0, a)^2$, and $\Omega_4 \equiv (0, a)^3$, and let the assumptions of Theorem 2.4 be true. Then for $2 \le |\alpha| \le 3$ the estimates*

$$\begin{aligned} \|D^\alpha u; L^p(\Omega_1)\|^p &\le C\varepsilon^{p(2-|\alpha|)-q} && \text{if } a \ge \tfrac{2p-1-q}{p}\tfrac{\varepsilon}{\gamma_0}\ln\tfrac{1}{\varepsilon},\quad q \ge 0, && (2.20)\\ \|D^\alpha u; L^p(\Omega_2)\|^p &\le C\varepsilon\left(\varepsilon^{1-|\alpha|}+\varepsilon^{-\alpha_3}\right)^p && \text{if } a \ge \tfrac{\varepsilon}{\gamma_0}\ln\tfrac{1}{\varepsilon}, && (2.21)\\ \|D^\alpha u; L^p(\Omega_3)\|^p &\le Ca\varepsilon\left(\varepsilon^{1-|\alpha|}+\varepsilon^{-(\alpha_2+\alpha_3)}\right)^p && \text{if } a \ge \tfrac{\varepsilon}{\gamma_0}\ln\tfrac{1}{\varepsilon}, && (2.22)\\ \|D^\alpha u; L^p(\Omega_4)\|^p &\le Ca^2\varepsilon^{1-p|\alpha|} && \forall a \in \left(0, \tfrac{1}{2}\right) && (2.23) \end{aligned}$$

hold. The estimation in each other subdomain is equivalent to one of the given cases.

Proof By integration. □

As to the knowledge of the authors, the results in Theorems 2.2 and 2.4 are (apart from the results in [14] for the Butuzov expansion, compare Section 1) the only available local estimates of single partial derivatives of the solution. Possibly, they are pessimistic. Or it may happen that under some more compatibility conditions to the data less restrictive local estimates may be proven.

Remark 2.1 In [22, pp. 15–18], the case of a smooth domain is considered. The consequence is that the terms $V_{\ell,m}$ and $V_{\ell,m,n}$ do not occur in (2.8) and (2.15), respectively. Moreover, the term $\varepsilon^{1-|\alpha|}$ does not occur in (2.10) and (2.17). The estimate is

$$|D^\alpha u(x)| \le M\left(1+\varepsilon^{2-|\alpha|}+\varepsilon^{-\alpha_d}e^{-\gamma_0 \operatorname{dist}(x,\partial\Omega)/\varepsilon}\right).$$

This led us to the conjecture that the term $\varepsilon^{1-|\alpha|}$ could be omitted in the estimate of $V_\ell(x)$ if the right hand side satisfies certain conditions in the neighbourhood of the non-smooth part of the boundary. The estimates (2.10) and (2.17) are then replaced by

$$|D^\alpha V_\ell(x)| \le M\varepsilon^{-\alpha_d}e^{-\gamma_0 x_{d,\ell}/\varepsilon}. \tag{2.24}$$

This assumption is similar to a result in [14, p. 407] where the case $c = \text{const.}$, $\Omega = (-1,1)^2$, was considered and compatibility conditions were extensively investigated.

This assumption (2.24) improves the Sobolev norm estimates. For $d = 2, 3$, we get

$$\|D^\alpha u; L^p(\Omega_2)\|^p \le C\left(a\varepsilon^{p(2-|\alpha|)}+\varepsilon^{1-p\alpha_d}\right) \quad \text{if } a \ge \tfrac{2p-1}{p}\tfrac{\varepsilon}{\gamma_0}\ln\tfrac{1}{\varepsilon}. \tag{2.25}$$

We will discuss in the next subsection that this assumption enhances the error estimates considerably.

Remark 2.2 For finite elements of degree k we have to consider $|\alpha| = k+1$ in Corollaries 2.3 and 2.5. First we remark that the cases $|\alpha| \geq 4$ are excluded. We can only assume that the assertions are valid also for higher derivatives. But the drawback for $|\alpha| = k+1 \geq 3$ is that even tangential derivatives degenerate strongly for $\varepsilon \to 0$. For the smooth part U we obtain $|D^\alpha U(x)| \leq M\varepsilon^{1-k}$, and in (2.18) we get for the tangential derivatives $|D^{(k+1,0,0)} V_{\ell,m}(x)| \leq M\varepsilon^{-k}$ close to the edges, which leads to error estimates which are severely non-uniform in ε.

In Theorem 2.8 we will consider instead of (2.9) and (2.16) the stronger condition

$$|D^\alpha U(x)| \leq M \tag{2.26}$$

and in place of (2.18) we will assume

$$|D^\alpha V_{\ell,m}(x)| \leq M\varepsilon^{-(\alpha_2+\alpha_3)} \min_{s=2,3} e^{-\gamma_0 x_{s,\ell,m}/\varepsilon} \tag{2.27}$$

Together with (2.24) we obtain in the three dimensional case for $|\alpha| \geq 2$

$$\begin{aligned}
\|D^\alpha u; L^2(\Omega_1)\|^2 &\leq C && \text{if } a \geq \tfrac{2|\alpha|-1}{2}\tfrac{\varepsilon}{\gamma_0}\ln\tfrac{1}{\varepsilon}, && (2.28)\\
\|D^\alpha u; L^2(\Omega_2)\|^2 &\leq C(\varepsilon^{1-2\alpha_d} + a) && \text{if } a \geq \tfrac{2|\alpha|-1}{2}\tfrac{\varepsilon}{\gamma_0}\ln\tfrac{1}{\varepsilon}, && (2.29)\\
\|D^\alpha u; L^2(\Omega_3)\|^2 &\leq C(a\varepsilon^{1-2(\alpha_2+\alpha_3)} + a^2) && \text{if } a \geq \tfrac{2|\alpha|-1}{2}\tfrac{\varepsilon}{\gamma_0}\ln\tfrac{1}{\varepsilon}, && (2.30)\\
\|D^\alpha u; L^2(\Omega_4)\|^2 &\leq Ca^2\varepsilon^{1-2|\alpha|} && \forall a \in \left(0,\tfrac{1}{2}\right). && (2.31)
\end{aligned}$$

In two dimensions, (2.28) and (2.29) hold then together with (2.14).

To show that the set of problems with such assumptions is not empty, consider

$$\begin{aligned}
-\varepsilon^2\Delta u + u &= \phi(x_1)\sin\pi x_2 + \phi(x_2)\sin\pi x_1 && \text{in } \Omega = (0,1)^2,\\
u &= 0 && \text{on } \partial\Omega,
\end{aligned}$$

$\phi(.)$ is an arbitrary, sufficiently smooth function. The solution of this problem is

$$u(x_1,x_2) = v(x_1)\sin\pi x_2 + v(x_2)\sin\pi x_1$$

where v is the solution of the one dimensional problem

$$\begin{aligned}
-\varepsilon^2 v'' + (1+\varepsilon^2\pi^2)\, v &= \phi && \text{in } (0,1),\\
v(0) = v(1) &= 0.
\end{aligned}$$

The splitting is $U = 0$, $V_{\ell,m} = 0$, $\ell, m = 1,\ldots,4$, $m > \ell$, $V_1 + V_3 = v(x_2)\sin\pi x_1$, $V_2 + V_4 = v(x_1)\sin\pi x_2$. The example can be extended to the three dimensional case in the obvious way.

2.3 Finite element error estimates

Using Theorem 2.1, the anisotropic local interpolation error estimate (2.7), and the localized Sobolev norm estimates of the solution u from Subsection 2.2, we can derive estimates for the energy norm of the finite element error. Consider first the two-dimensional case ($d = 2$) and linear elements ($k = 1$) and remember that

$$\begin{aligned}
h_{1,e} = h_{2,e} &= (1-2a)h && \text{in } \Omega_1,\\
h_{1,e} = (1-2a)h, \quad h_{2,e} &= ah && \text{in } \Omega_2,\\
h_{1,e} = h_{2,e} &= ah && \text{in } \Omega_3.
\end{aligned} \tag{2.32}$$

Theorem 2.6 *Let the assumptions of Theorem 2.2 be valid. For $d = 2$, $k = 1$, any $\gamma_0 \in (0, \gamma)$, $a = a_0 \frac{\varepsilon}{\gamma_0} \ln \frac{1}{\varepsilon}$, $a_0 \geq 1$, the estimate*

$$\||\, u - u_h \,\|| \leq Ch \left(\varepsilon^{1/2} \ln \tfrac{1}{\varepsilon} + \varepsilon^{-1/2} h \right) \tag{2.33}$$

holds. If we assume that (2.24) holds instead of (2.10) then the estimate improves to

$$\||\, u - u_h \,\|| \leq Ch \left(\varepsilon^{1/2} \ln \tfrac{1}{\varepsilon} + h \right) \tag{2.34}$$

if $a_0 \geq \frac{3}{2}$. For γ see (H.2).

Proof Because of $d = 2$ we obtain from (2.5) and (2.7)

$$\begin{aligned} \||\, u - I_h^{(1)} u \,\||_e^2 &\leq \|c; L^\infty(e)\| \|u - I_h^{(1)} u; L^2(e)\|^2 + \varepsilon^2 |u - I_h^{(1)} u; W^{1,2}(e)|^2 \\ &\leq C \sum_{|\alpha|=1} \sum_{|\beta|=1} (h^{2(\alpha+\beta)} + \varepsilon^2 h^{2\alpha}) \|D^{\alpha+\beta} u; L^2(e)\|^2. \end{aligned}$$

Using (2.32) and Corollary 2.3 we get

$$\begin{aligned} \||\, u - I_h^{(1)} u \,\||_{\Omega_1}^2 &\leq C(h^4 + \varepsilon^2 h^2)\varepsilon^{-1} \;=\; Ch^2(\varepsilon + \varepsilon^{-1} h^2), \\ \||\, u - I_h^{(1)} u \,\||_{\Omega_2}^2 &\leq C[(h^4 + \varepsilon^2 h^2)\varepsilon^{-1} + (a^2 h^4 + \varepsilon^2 a h^2)\varepsilon^{-1} + (a^4 h^4 + \varepsilon^2 a^2 h^2)\varepsilon^{-3}] \\ &\leq Ch^2 \left(\varepsilon (\ln \tfrac{1}{\varepsilon})^2 + \varepsilon^{-1} h^2 \right), \\ \||\, u - I_h^{(1)} u \,\||_{\Omega_3}^2 &\leq C(a^4 h^4 + \varepsilon^2 a^2 h^2)\varepsilon^{-3} a \\ &\leq Ch^2 \left(\varepsilon^2 (\ln \tfrac{1}{\varepsilon})^3 + \varepsilon^2 (\ln \tfrac{1}{\varepsilon})^5 h^2 \right). \end{aligned}$$

Because all other subdomains of Ω are equivalent to one of these cases we conclude

$$\||\, u - u_h \,\||^2 \leq \||\, u - I_h^{(1)} u \,\||^2 \leq Ch^2 \left(\varepsilon (\ln \tfrac{1}{\varepsilon})^2 + \varepsilon^{-1} h^2 \right).$$

In the case of Remark 2.1 we obtain

$$\begin{aligned} \||\, u - I_h^{(1)} u \,\||_{\Omega_2}^2 &\leq C[(h^4 + \varepsilon^2 h^2)(\varepsilon + a) + (a^2 h^4 + \varepsilon^2 a h^2)\varepsilon^{-1} + (a^4 h^4 + \varepsilon^2 a^2 h^2)\varepsilon^{-3}] \\ &\leq Ch^2 \left(\varepsilon (\ln \tfrac{1}{\varepsilon})^2 + \varepsilon (\ln \tfrac{1}{\varepsilon})^4 h^2 \right). \end{aligned}$$

To eliminate the negative power of ε in the case Ω_1 we use (2.12) with $q = 0$:

$$\||\, u - I_h^{(1)} u \,\||_{\Omega_1}^2 \leq C(h^4 + \varepsilon^2 h^2).$$

The assertion follows with the same arguments as above. □

Remark 2.3 By following the proof one can observe that a can be increased while keeping the approximation order. We get for $\frac{\varepsilon}{\gamma_0} \ln \frac{1}{\varepsilon} \leq a \leq C\varepsilon^{1/2}$ the estimate $\||\, u - u_h \,\|| \leq Ch(1 + \varepsilon^{-1/2} h)$, and under the assumption made in Remark 2.1 we obtain $\||\, u - u_h \,\|| \leq Ch(h + \varepsilon^{1/4})$ if $\frac{3}{2} \frac{\varepsilon}{\gamma_0} \ln \frac{1}{\varepsilon} \leq a \leq C\varepsilon^{3/4}$. Note that the dependence on ε is then less favourable.

The three-dimensional case can be treated with similar arguments. The main difference is that the local interpolation error estimate (2.7) does not hold for $m = k = 1$, $p = 2$. We circumvent this problem by using some $p > 2$. As introduced in Subsection 2.1 we set

$$\begin{aligned} h_{1,e} = h_{2,e} = h_{3,e} &= (1 - 2a)h && \text{in } \Omega_1, \\ h_{1,e} = h_{2,e} = (1 - 2a)h, \quad h_{3,e} &= ah && \text{in } \Omega_2, \\ h_{1,e} = (1 - 2a)h, \quad h_{2,e} = h_{3,e} &= ah && \text{in } \Omega_3, \\ h_{1,e} = h_{2,e} = h_{3,e} &= ah && \text{in } \Omega_4. \end{aligned} \tag{2.35}$$

Theorem 2.7 *Let the assumptions of Theorem 2.4 be valid. For $d=3$, $k=1$, some fixed $p>2$, any $\gamma_0 \in (0,\gamma)$ $a = a_0 \frac{\varepsilon}{\gamma_0} \ln \frac{1}{\varepsilon}$, $a_0 \geq 1$, the estimate*

$$|||\, u - u_h \,||| \leq Ch \left(\varepsilon^{1/2} (\ln \tfrac{1}{\varepsilon})^{3/2-1/p} + \varepsilon^{-1/2} h \right) \tag{2.36}$$

holds. If we assume that (2.24) holds instead of (2.17) then the estimate improves to

$$|||\, u - u_h \,||| \leq Ch \left(\varepsilon^{1/2} (\ln \tfrac{1}{\varepsilon})^{3/2-1/p} + h \right),$$

if $a_0 \geq \frac{3}{2}$ arbitrary. For γ see (H.2).

Proof In Ω_1 and Ω_4 we have isotropic elements. Thus there hold with (2.35) the relations

$$\begin{aligned} |||\, u - I_h^{(1)} u \,|||_{\Omega_1}^2 &\leq C(h^4 + \varepsilon^2 h^2) \varepsilon^{-3+2a_1}, \qquad a_1 \equiv \min\{3; 2a_0\}, &(2.37)\\ |||\, u - I_h^{(1)} u \,|||_{\Omega_4}^2 &\leq C(a^4 h^4 + \varepsilon^2 a^2 h^2) \varepsilon^{-3} a^2 \\ &\leq Ch^2 \left(\varepsilon^3 (\ln \tfrac{1}{\varepsilon})^4 + \varepsilon^3 (\ln \tfrac{1}{\varepsilon})^6 h^2 \right). &(2.38) \end{aligned}$$

For Ω_i, $i = 2,3$, we use the Hölder inequality and (2.7) to get

$$\begin{aligned} |||\, u - I_h^{(1)} u \,|||_{\Omega_i}^2 &\leq \|c; L^\infty(\Omega_i)\| \|u - I_h^{(1)} u; L^2(\Omega_i)\|^2 + \\ &\quad + (\mathrm{meas}\Omega_i)^{1-2/p} \varepsilon^2 |u - I_h^{(1)} u; W^{1,p}(\Omega_i)|^2 \\ &\leq C \sum_{|\alpha|=1} \sum_{|\beta|=1} h^{2(\alpha+\beta)} \|D^{\alpha+\beta} u; L^2(\Omega_i)\|^2 + \\ &\quad + (\mathrm{meas}\Omega_i)^{1-2/p} \varepsilon^2 \sum_{|\alpha|=1} \sum_{|\beta|=1} h^{2\alpha} \|D^{\alpha+\beta} u; L^p(\Omega_i)\|^2. \end{aligned}$$

Now we proceed as in the proof of Theorem 2.6. Using (2.35) and Corollary 2.5 we get

$$\begin{aligned} |||\, u - I_h^{(1)} u \,|||_{\Omega_2}^2 &\leq C(h^4 \varepsilon^{-1} + a^4 h^4 \varepsilon^{-3}) + Ca^{1-2/p} \varepsilon^2 (\varepsilon^{-2+2/p} h^2 + \varepsilon^{-4+2/p} a^2 h^2) \\ &\leq Ch^4 (\varepsilon^{-1} + a^4 \varepsilon^{-3}) + Ch^2 (a^{1-2/p} \varepsilon^{2/p} + a^{3-2/p} \varepsilon^{-2+2/p}) \\ &\leq Ch^2 \left(\varepsilon (\ln \tfrac{1}{\varepsilon})^{3-2/p} + h^2 \varepsilon^{-1} \right) \\ |||\, u - I_h^{(1)} u \,|||_{\Omega_3}^2 &\leq C(h^4 a \varepsilon^{-1} + a^4 h^4 a \varepsilon^{-3}) + \\ &\quad + Ca^{2(1-2/p)} \varepsilon^2 (\varepsilon^{-2+2/p} a^{2/p} h^2 + \varepsilon^{-4+2/p} a^{2+2/p} h^2) \\ &\leq Ch^4 (a\varepsilon^{-1} + a^5 \varepsilon^{-3}) + Ch^2 (a^{2-2/p} \varepsilon^{2/p} + a^{4-2/p} \varepsilon^{-2+2/p}) \\ &\leq Ch^2 \left(\varepsilon^2 (\ln \tfrac{1}{\varepsilon})^{4-2/p} + h^2 \ln \tfrac{1}{\varepsilon} \right). &(2.39) \end{aligned}$$

Thus for $a_0 \geq 1$

$$|||\, u - I_h^{(1)} u \,|||^2 \leq Ch^2 \left(\varepsilon (\ln \tfrac{1}{\varepsilon})^{3-2/p} + \varepsilon^{-1} h^2 \right).$$

With (2.25) instead of (2.21) we obtain

$$\begin{aligned} |||\, u - I_h^{(1)} u \,|||_{\Omega_2}^2 &\leq C[h^4 (a + \varepsilon) + a^2 h^4 \varepsilon^{-1} + a^4 h^4 \varepsilon^{-3}] + \\ &\quad + Ca^{1-2/p} \varepsilon^2 [h^2 (a+\varepsilon)^{2/p} + \varepsilon^{-2+2/p} h^2 + \varepsilon^{-4+2/p} a^2 h^2] \\ &\leq Ch^4 (a + a^2 \varepsilon^{-1} + a^4 \varepsilon^{-3}) + Ch^2 (a\varepsilon^2 + a^{1-2/p} \varepsilon^{2/p} + a^{3-2/p} \varepsilon^{-2+2/p}) \\ &\leq Ch^2 \left(\varepsilon (\ln \tfrac{1}{\varepsilon})^{3-2/p} + \varepsilon (\ln \tfrac{1}{\varepsilon})^4 h^2 \right). \end{aligned}$$

Together with (2.37) ($a_0 \geq \frac{3}{2}$), (2.38) and (2.39) we conclude the second part of the assertion.

□

Remark 2.4 Again, a can be increased while keeping the approximation order. We get for $\frac{\varepsilon}{\gamma_0}\ln\frac{1}{\varepsilon} \le a \le C\varepsilon^{(2-2/p)/(3-2/p)}$ the estimate $|||\, u - u_h \,||| \le Ch(1+\varepsilon^{-1/2}h)$. Under the assumption made in Remark 2.1 we obtain in the case of $a = \varepsilon^s$, $0 < s < 1$, an estimate with a negative power of ε as well.

Consider now shape functions of degree $k \ge 2$. In the case of assumptions as in Theorems 2.2 and 2.4 the situation is unsatisfactory: We obtain estimates not better than

$$|||\, u - u_h \,||| \le Ch^k(\varepsilon + h)\varepsilon^{1-k}$$

already because of the estimates (2.9) and (2.20) in the domain Ω_1 without layers. However, we get error estimates uniform with respect to ε for $k \ge 2$ in the case of the strong assumptions discussed in Remark 2.2.

Theorem 2.8 *For $d = 2$ assume that the exact solution of (1.1), (1.2) satisfies the assumptions (2.8), (2.11), (2.24), and (2.26). For $d = 3$ assume (2.8), (2.19), (2.24), (2.26), and (2.27). Then the error of the Galerkin finite element solution satisfies*

$$|||\, u - u_h \,||| \le Ch^k(h + \varepsilon^{1/(2k+2)}) \tag{2.40}$$

if $\frac{2k+1}{2}\frac{\varepsilon}{\gamma_0}\ln\frac{1}{\varepsilon} \le a \le \varepsilon^{(2k+1)/(2k+2)}$, $k \ge 2$. In the special case $a = a_0\frac{\varepsilon}{\gamma_0}\ln\frac{1}{\varepsilon}$, $a_0 \ge \frac{2k+1}{2}$, the estimate can be sharpened to

$$|||\, u - u_h \,||| \le Ch^k\left(\varepsilon^{1/2}(\ln\tfrac{1}{\varepsilon})^k + h\right). \tag{2.41}$$

Proof For $k \ge 2$ we obtain from (2.5) and (2.7) the estimate

$$|||\, u - I_h^{(k)}u \,|||_e^2 \le C \sum_{|\alpha|=k}\sum_{|\beta|=1}\left(h_e^{2(\alpha+\beta)} + \varepsilon^2 h_e^{2\alpha}\right)\|D^{\alpha+\beta}u; L^p(e)\|^2$$

in both the two and the three dimensional case.

In the two dimensional case we use (2.32) and the Sobolev norm estimates (2.14), (2.28), and (2.29) and obtain

$$\begin{aligned}
|||\, u - I_h^{(k)}u \,|||_{\Omega_1}^2 &\le Ch^{2k}(\varepsilon^2 + h^2) && \text{for } a \ge \tfrac{3}{2}\tfrac{\varepsilon}{\gamma_0}\ln\tfrac{1}{\varepsilon},\\
|||\, u - I_h^{(k)}u \,|||_{\Omega_2}^2 &\le C\sum_{|\alpha|=k}\sum_{|\beta|=1}\left(a^{2(\alpha_2+\beta_2)}h^{2(k+1)} + \varepsilon^2 a^{2\alpha_2}h^{2k}\right)\left(\varepsilon^{1-2(\alpha_2+\beta_2)} + a\right) &&\\
&\le Ch^{2k}\left(a^{2(k+1)}\varepsilon^{1-2(k+1)}h^2 + a^{2k}\varepsilon^{1-2k}\right) &&\\
&\le Ch^{2k}\left(\varepsilon^{1/(k+1)} + h^2\right) && \text{for } a \le \varepsilon^{(2k+1)/(2k+2)},\\
|||\, u - I_h^{(k)}u \,|||_{\Omega_3}^2 &\le C(ah)^{2k}\left(\varepsilon^2 + (ah)^2\right)\, a\varepsilon^{1-2(k+1)} &&\\
&= Ch^{2k}\left(\varepsilon^{1-2k}a^{1+2k} + \varepsilon^{-1-2k}a^{3+2k}h^2\right) &&\\
&\le Ch^{2k}\left(\varepsilon^{4/(2k+3)} + h^2\right) && \text{for } a \le \varepsilon^{(2k+1)/(2k+3)}.
\end{aligned}$$

Thus (2.40) holds.

In the three dimensional case we apply (2.35) and the Sobolev norm estimates (2.28)–(2.31) and obtain

$$\begin{aligned}
|||\, u - I_h^{(k)}u \,|||_{\Omega_1}^2 &\le Ch^{2k}(\varepsilon^2 + h^2) && \text{for } a \ge \tfrac{3}{2}\tfrac{\varepsilon}{\gamma_0}\ln\tfrac{1}{\varepsilon},\\
|||\, u - I_h^{(k)}u \,|||_{\Omega_2}^2 &\le C\sum_{|\alpha|=k}\sum_{|\beta|=1}\left(a^{2(\alpha_3+\beta_3)}h^{2(k+1)} + \varepsilon^2 a^{2\alpha_3}h^{2k}\right)\left(\varepsilon^{1-2(\alpha_3+\beta_3)} + a\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq Ch^{2k}\left(\varepsilon^{1/(k+1)}+h^2\right) && \text{for } a\leq \varepsilon^{(2k+1)/(2k+2)},\\
\|\!| u-I_h^{(k)}u \|\!|^2_{\Omega_3} &\leq C\sum_{|\alpha|=k}\sum_{|\beta|=1}\left(a^{2(\alpha_2+\alpha_3+\beta_2+\beta_3)}h^{2(k+1)}+\varepsilon^2 a^{2(\alpha_2+\alpha_3)}h^{2k}\right)\times\\
&\qquad\qquad\times\left(a\varepsilon^{1-2(\alpha_2+\alpha_3+\beta_2+\beta_3)}+a^2\right)\\
&\leq Ch^{2k}\left(a^{1+2(k+1)}\varepsilon^{1-2(k+1)}h^2+a^{1+2k}\varepsilon^{1-2k}\right)\\
&\leq Ch^{2k}\left(\varepsilon^{4/(2k+3)}+h^2\right) && \text{for } a\leq \varepsilon^{(2k+1)/(2k+3)},\\
\|\!| u-I_h^{(k)}u \|\!|^2_{\Omega_4} &\leq C(ah)^{2k}\left(\varepsilon^2+(ah)^2\right)\, a^2\varepsilon^{1-2(k+1)}\\
&= Ch^{2k}\left(\varepsilon^{1-2k}a^{2+2k}+\varepsilon^{-1-2k}a^{4+2k}h^2\right)\\
&\leq Ch^{2k}\left(\varepsilon^{3/(k+2)}+h^2\right) && \text{for } a\leq \varepsilon^{(2k+1)/(2k+4)}.
\end{aligned}
$$

With these estimates (2.40) is concluded. By setting $a=\mathcal{O}(\varepsilon\ln\frac{1}{\varepsilon})$ in these estimates we get (2.41). □

Concluding this subsection we can state that the three differently strong assumptions in Subsection 2.2 lead to different error estimates. The weakest assumptions were covered by Theorems 2.2 and 2.4. In this case we get for linear elements and $a=\frac{\varepsilon}{\gamma_0}\ln\frac{1}{\varepsilon}$ the error estimate

$$\|\!| u-u_h \|\!| \leq Ch\left(\varepsilon^{1/2-\delta}+\varepsilon^{-1/2}h\right),$$

$\delta>0$ arbitrarily small. For the slightly stronger assumptions discussed in Remark 2.1 and with $a=\frac{3}{2}\frac{\varepsilon}{\gamma_0}\ln\frac{1}{\varepsilon}$ we obtain for linear elements

$$\|\!| u-u_h \|\!| \leq Ch\left(\varepsilon^{1/2-\delta}+h\right).$$

For elements of higher degree $k\geq 2$ we need still stronger assumptions as introduced in Remark 2.2 to obtain

$$\|\!| u-u_h \|\!| \leq Ch^k\left(\varepsilon^{1/2-\delta}+h\right).$$

All these estimates hold for $d=2,3$, and the number of elements is of the order h^{-d} independent of ε.

2.4 Numerical test

As an example we took the boundary value problem from [20, Example 11.3]:

$$
\begin{aligned}
-\varepsilon^2\Delta u+u &= 0 && \text{in } \Omega=(0,1)^2,\\
u &= e^{-x_1/\varepsilon}+e^{-x_2/\varepsilon} && \text{on } \partial\Omega.
\end{aligned}
$$

Actually the inhomogeneity is in the boundary condition but not in the differential equation. However, it was the only problem with known exact solution we found calculated by other authors.

The problem has a boundary layer only at $M=\{x\in\partial\Omega : x_1=0\vee x_2=0\}$. Therefore we use a domain decomposition into four rectangles $(0,a)^2$, $(0,a)\times(a,1)$, $(a,1)\times(0,a)$, and $(a,1)^2$. The rectangles were uniformly hierarchically refined as described in Subsection 2.1, see Figure 2.2.

In order to investigate the influence of anisotropic mesh refinement on the approximation we varied the mesh size h and computed numerical solutions for different values of ε and a. From them we calculated the energy norm $\|\!| u-u_h \|\!|$ of the finite element error by a numerical integration formula which was determined such that the integration error was independent of h (but dependent on $u(\varepsilon)$ and a). The error is given in Tables 2.1–2.4.

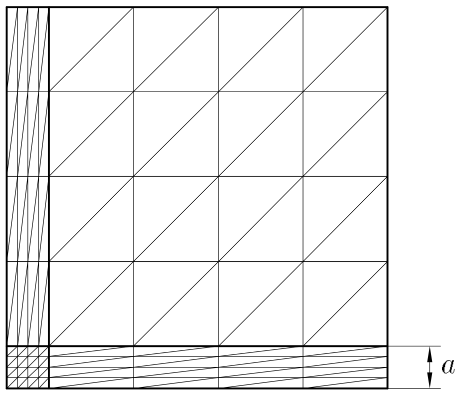

Figure 2.2: Anisotropically refined mesh for the numerical test, $h = \frac{1}{4}$.

h^{-1}	$\varepsilon = 10^{-1}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-5}$
4	0.114 e−0	0.278 e−0	0.282 e−0
8	0.570 e−1	0.189 e−0	0.195 e−0
16	0.285 e−1	0.128 e−0	0.136 e−0
32	0.143 e−1	0.856 e−1	0.955 e−1
64	0.713 e−2	0.543 e−1	0.674 e−1

Table 2.1: Error norm for $a = 0.5$.

h^{-1}	$\varepsilon = 10^{-1}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-5}$
4	0.747 e−1	0.894 e−2	0.130 e−2
8	0.387 e−1	0.518 e−2	0.657 e−3
16	0.196 e−1	0.362 e−2	0.330 e−3
32	0.980 e−2	0.298 e−2	0.167 e−3
64	0.490 e−2	0.256 e−2	0.877 e−4

Table 2.2: Error norm for $a = \varepsilon \log_{10} \frac{1}{\varepsilon}$.

h^{-1}	$\varepsilon = 10^{-1}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-5}$
4	0.511 e−1	0.134 e−1	0.218 e−2
8	0.257 e−1	0.681 e−2	0.112 e−2
16	0.129 e−1	0.342 e−2	0.568 e−3
32	0.644 e−2	0.171 e−2	0.285 e−3
64	0.322 e−2	0.864 e−3	0.143 e−3

Table 2.3: Error norm for $a = 2\varepsilon \log_{10} \frac{1}{\varepsilon}$.

h^{-1}	$\varepsilon = 10^{-1}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-5}$
4	0.912 e−1	0.257 e−1	0.395 e−2
8	0.456 e−1	0.134 e−2	0.217 e−2
16	0.228 e−1	0.680 e−2	0.112 e−3
32	0.114 e−1	0.342 e−2	0.568 e−3
64	0.571 e−2	0.171 e−2	0.285 e−3

Table 2.4: Error norm for $a = 4\varepsilon \log_{10} \frac{1}{\varepsilon}$.

h^{-1}	$\varepsilon = 10^{-1}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-5}$
4	0.162 e−0	0.141 e−0	0.138 e−0
8	0.813 e−1	0.718 e−1	0.708 e−1
16	0.408 e−1	0.360 e−1	0.359 e−1
32	0.204 e−1	0.180 e−1	0.180 e−1
64	0.102 e−1	0.911 e−2	0.904 e−2

Table 2.5: Scaled error norm $|\!|\!| u - u_h |\!|\!| / (\varepsilon^{1/2} \log_{10} \frac{1}{\varepsilon})$ for $a = 2\varepsilon \log_{10} \frac{1}{\varepsilon}$.

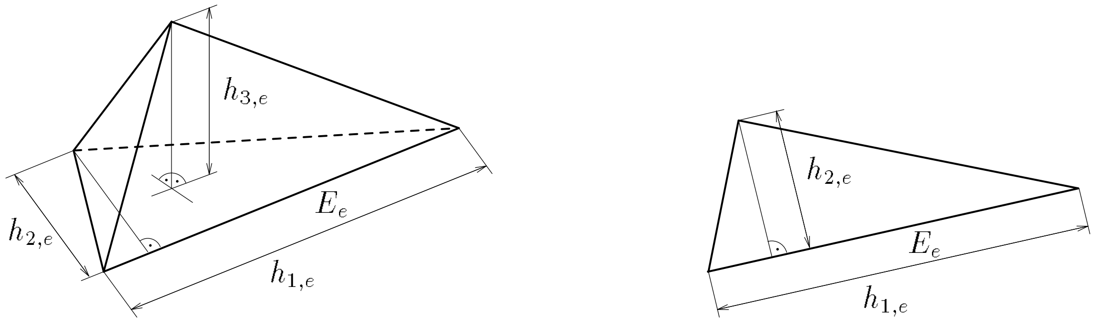

Figure 3.1: Element related mesh sizes.

In Table 2.1 the error is displayed when a quasiuniform mesh is used. We see a good asymptotics of the error in the case of a large value of ε, but the error is far from this asymptotics in case of small ε. We remark that the values for $\varepsilon = 10^{-5}$ may be incorrect because the numerical integration may not have resolved the layer correctly.

In the case of $a = \mathcal{O}(\varepsilon \ln \frac{1}{\varepsilon})$ we obtain the expected order of the approximation error for small ε as well. Moreover, we validate the theoretical statement (2.34) that the error is diminishing with decreasing ε, see Table 2.5. (The term Ch^2 is neglected.) Comparing Tables 2.2–2.4 we can see the influence of the linear scaling of a by the parameter a_0. If a_0 is chosen too large or too small then the error is increasing. From this test we can conjecture that the optimal a_0 is dependent on ε in a nonlinear manner.

3 Anisotropic local interpolation error estimates

3.1 Current state

Consider an anisotropic simplicial element $e \subset \mathbb{R}^d$, $d = 2, 3$, with sizes $h_{1,e}, \dots, h_{d,e}$, as introduced in Figure 3.1. Moreover, we introduce the usual Lagrangian interpolation operator $I_h^{(k)} : \mathcal{C}(\overline{e}) \to \mathcal{P}_k(e)$, where $\mathcal{P}_k$ is again the space of polynomials of maximal degree $k \geq 1$. $\mathcal{C}(\overline{e})$ is the set of all functions which are continuous in $\overline{e}$. It was proved in [1] that, under certain conditions to the geometry as discussed below, the interpolation error can be estimated by

$$|v - I_h^{(k)} v; W^{1,p}(e)|^p \leq C \sum_{|\alpha|=k} h_e^{\alpha p} |D^\alpha v; W^{1,p}(e)|^p \tag{3.1}$$

provided that $d = 2$ or $k > 1$ or $p > 2$. For our application in Section 4 we need an estimate of the interpolation error in $W^{2,p}(e)$. Thus we shall generalize (3.1) and prove for $m = 0, \dots, k$

$$|v - I_h^{(k)} v; W^{m,p}(e)|^p \leq C \sum_{|\alpha|=k+1-m} h_e^{\alpha p} |D^\alpha v; W^{m,p}(e)|^p, \tag{3.2}$$

which holds if $d = 2$ or $m < k$ or $p > 2$.

In [1], the assumptions on the geometry of the element were formulated in two dimensions by the following two conditions:

Maximal angle condition (2D): There is a constant $\gamma_* < \pi$ (independent of h and $e \in \mathcal{T}_h$) such that the maximal interior angle γ_e of any element e is bounded by $\gamma_* :\ \gamma_e \leq \gamma_*$.

Coordinate system condition (2D): The angle ψ_e between the longest side of the element e and the x_1-axis is bounded by $|\sin \psi_e| \leq C h_{2,e}/h_{1,e}$.

The 3D-counterpart of the maximal angle condition was formulated in [1, page 290] rather abstract and with a misprint. It reads correctly:

$$\min_{i=1,\ldots,3}\max_{j=1,\ldots,6}|(b_j,e_i)|\geq C_0>0,$$

where e_i ($i=1,\ldots,3$) denotes the i-th unit vector of the coordinate system and b_j ($j=1,\ldots,6$) are the directions of edges of the simplex e. The 3D-counterpart of the coordinate system condition was not elaborated at all. Because this is very unsatisfactory, we formulate here 3D-versions of these two conditions in a geometric way as in the two-dimensional case, see Section 3.3.

3.2 Estimates on the reference element

For the proof of the error estimates we proceed in the usual way: (1) transformation of the left-hand side to some reference element $\hat{e}$, (2) estimation of the error on the reference element $\hat{e}$, (3) transformation of the right-hand side to the element e. We recall that the transformation is done to get estimates with powers of h and a constant which is independent of the actual element. Hence, we can also use a finite number of reference elements. The choice of appropriate elements $\hat{e}$ is discussed in Subsection 3.3. Each reference element has the following property (P). We will use it in the proof of Theorem 3.1 (error estimation on $\hat{e}$).

Property (P) For each axis of the coordinate system $(y_1,\ldots,y_d)$ there is one edge of $\hat{e}\subset I\!R^d$, $d=2,3$, that has length one and is parallel to this axis.

Theorem 3.1 *Let $\hat{e}\subset I\!R^d$, $d=2,3$, be a reference element with property (P), and let $I^{(k)}\hat{v}$ be the Lagrangian interpolant of $\hat{v}\in W^{k+1,p}(\hat{e})$ with polynomials of order k. Then for any multi-index γ with $|\gamma|\leq k$ the estimate*

$$\|D^\gamma(\hat{v}-I^{(k)}\hat{v});L^p(\hat{e})\|\leq C|D^\gamma\hat{v};W^{k+1-|\gamma|,p}(\hat{e})| \tag{3.3}$$

holds if and only if $d=2$ or $\gamma\notin\{(k,0,\ldots,0),\ldots,(0,\ldots,0,k)\}$ or $p>2$.

Note that estimates as in (3.3), with a seminorm of $D^\gamma\hat{v}$ at the right hand side, are necessary to get anisotropic estimates, see the introductory example in [1]. We remark further, that (3.3) was proved in [16] in another way than we do here. The disadvantage of the approach in [16] is that m-th derivatives of v are required to be continuous, that means a stronger assumption $k+1-m>n/p$ there. Finally, we mention that interpolation error estimates for anisotropic elements were proved in [5] as well. These authors use similar ideas on a less formal level than we do here. They followed their ideas only in the special case $d=2$ and $m=1$.

Proof We proceed in analogy to the proof of Theorem 1 in [1], where $|\gamma|=1$ is assumed. We use Lemma 3 of that paper with $P=\mathcal{P}_k$, $Q=\mathcal{P}_{k-|\gamma|}$, that means, it remains to find linear functionals $f_i\in(W^{k+1-|\gamma|,p}(\hat{e}))'$, $i=1,\ldots,J$, $J=\dim\mathcal{P}_{k-|\gamma|}=\binom{k-|\gamma|+d}{d}$, with the properties

$$f_i(D^\gamma I^{(k)}\hat{v})=f_i(D^\gamma\hat{v}),\quad i=1,\ldots,J,\quad \textit{for all } \hat{v}\in W^{k+1,p}(\hat{e}), \tag{3.4}$$

$$\textit{if all } f_i,\ i=1,\ldots,J,\ \textit{vanish on some } q\in\mathcal{P}_{k-|\gamma|},\ \textit{then } q=0. \tag{3.5}$$

We will illustrate this choice in four typical examples, all other cases are then canonical. In all cases one can prove (3.4) owing to $\hat{v}(y)=I^{(k)}\hat{v}(y)$ in the nodal points. For the

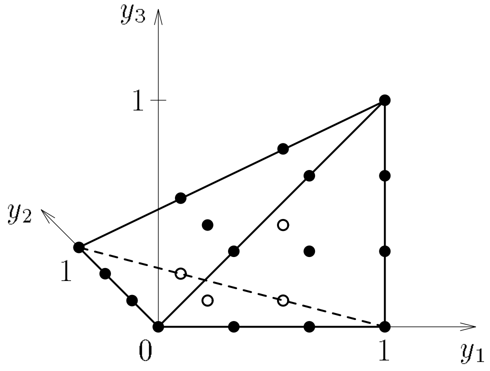

Figure 3.2: Nodes for a cubic tetrahedral element.

illustration we choose the reference tetrahedron $\hat{e}$ with the vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(1,0,1)$, and $k=3$, see Figure 3.2. A cubic element is chosen because all four cases can be explained only for $k \geq 3$.

(a) For $\gamma = (2,0,0)$ we have $J = \dim \mathcal{P}_1 = 4$ and we choose

$$f_i(w) = \int_{y_1^{(i)}}^{y_1^{(i)}+\frac{1}{3}} \int_{\xi}^{\frac{1}{3}+\xi} w(y_1, y_2^{(i)}, y_3^{(i)})\, dy_1 d\xi, \quad i = 1, \ldots, 4,$$

with $y^{(1)} = (0,0,0)$, $y^{(2)} = (\frac{1}{3},0,0)$, $y^{(3)} = (0,\frac{1}{3},0)$, and $y^{(4)} = (\frac{1}{3},0,\frac{1}{3})$. Property (3.5) is easily checked. Due to trace theorems we have $|f_i(w)| \leq C\|w; W^{2,p}(\hat{e})\|$ for any $p \geq 1$, and all desired properties are proved.

(b) For $\gamma = (1,1,0)$ we choose

$$f_i(w) = \int_{y_1^{(i)}}^{y_1^{(i)}+\frac{1}{3}} \int_{y_2^{(i)}}^{y_2^{(i)}+\frac{1}{3}} w(y_1, y_2, y_3^{(i)})\, dy_2 dy_1, \quad i = 1, \ldots, 4,$$

with $y^{(i)}$ as in (a) and proceed as above.

(c) For $\gamma = (3,0,0)$ we have $J = 1$ and choose

$$f(w) = \int_0^{\frac{1}{3}} \int_{\xi}^{\frac{1}{3}+\xi} \int_{\eta}^{\frac{1}{3}+\eta} w(y_1,0,0)\ dy_1 d\eta d\xi.$$

The main difference to (a) is that this functional is bounded in $W^{1,p}(\hat{e})$ only for $p > 2$. The proof of the reverse direction, namely that (3.3) does not hold for $p \leq 2$, is carried out by a slight modification of the counterexample in [1, page 283]; we have to use $v_\varepsilon(y) = (1 - \min\{1;\ \varepsilon \ln|\ln(r/e)|\})\, y_1^3$, $r = (y_2^2 + y_3^2)^{1/2}$.

(d) For $\gamma = (2,1,0)$ we choose

$$f(w) = \int_0^{\frac{1}{3}} \int_{\xi}^{\frac{1}{3}+\xi} \int_0^{\frac{1}{3}} w(y_1, y_2, 0)\, dy_2 dy_1 d\xi$$

and find that this functional is bounded in $W^{1,p}(\hat{e})$ for all $p \geq 1$. □

3.3 Coordinate transformation

The aim of this subsection is to investigate the transformation of estimate (3.3) from a finite number of reference elements $\hat{e}$ to the element e. We recall that such a transformation can be realized by

$$x = F(y) = By + b \tag{3.6}$$

with $B \in I\!R^{d\times d}$, $b \in I\!R^d$, $d = 2,3$, $e = F(\hat{e})$; $y = (y_1,\dots,y_d)$ is the coordinate system of the reference element $\hat{e}$, and $x = (x_1,\dots,x_d)$ is the system which our problem is considered in (possibly adapted to the domain or data, but independent of the discretization). For intermediate use we introduce another Cartesian coordinate system $(x_{1,e}, x_{2,e}, x_{3,e})$ (related to the element e) such that $(0,0,0)$ is a vertex of e, and the longest edge E_e is part of the $x_{1,e}$-axis. In three dimensions we require additionally that the larger of the two faces of e which contain E_e, is part of the $x_{1,e}, x_{2,e}$-plane.

We are now ready to formulate the three-dimensional equivalents of the maximal angle condition and the coordinate system condition, compare Subsection 3.1 for the two-dimensional ones.

Maximal angle condition (3D): There is a constant $\gamma_* < \pi$ (independent of h and $e \in \mathcal{T}_h$) such that the maximal interior angle $\gamma_{F,e}$ of the four faces as well as the maximal angle $\gamma_{E,e}$ between two faces of any element e is bounded by γ_* : $\gamma_{F,e} \le \gamma_*$, $\gamma_{E,e} \le \gamma_*$.

Coordinate system condition (3D): The transformation of the element related coordinate system $(x_{1,e}, x_{2,e}, x_{3,e})$ into the discretization independent system (x_1, x_2, x_3) can be determined as a translation and three rotations around the $x_{j,e}$-axes by angles $\psi_{j,e}$ $(j = 1,2,3)$, where

$$|\sin\psi_{1,e}| \le Ch_{3,e}/h_{2,e}, \quad |\sin\psi_{2,e}| \le Ch_{3,e}/h_{1,e}, \quad |\sin\psi_{3,e}| \le Ch_{2,e}/h_{1,e}. \tag{3.7}$$

These conditions yield properties of the transformation matrix B from (3.6) which are sufficient for our anisotropic interpolation error estimates.

Lemma 3.2 *For each element e, one can choose a reference element with property (P) such that the elements of the matrix B satisfy the following relations.*

$$\begin{aligned} |b_{ji}| &\le C \min\{h_{j,e}, h_{i,e}\}, \quad j,i = 1,\dots,d, \\ \left|b_{ji}^{(-1)}\right| &\le C \min\{h_{j,e}^{-1}, h_{i,e}^{-1}\}, \quad j,i = 1,\dots,d, \end{aligned}$$

Here, b_{ji} are the elements of B, and $b_{ji}^{(-1)}$ are those of B^{-1}.

In two dimensions the reference element can be chosen as usual, in three dimensions we use two reference elements, see Figure 3.3. Note that anisotropic tetrahedra can have three or four edges with length of order $h_{1,e}$. They are mapped to $\hat{e}_1$ and $\hat{e}_2$, respectively. (In the case of 5 edges with length of order $h_{1,e}$ either element can be used.) — The proof of this lemma is rather lengthy and technical and is omitted here. It can be found in [2].

Theorem 3.3 *Assume that the element e satisfies the maximal angle condition and the coordinate system condition. Then for $v \in W^{k+1,p}(e)$, $I_h^{(k)}v \in \mathcal{P}_k(e)$ and $m = 0,\dots,k$, the estimate*

$$|v - I_h^{(k)}v; W^{m,p}(e)|^p \le C \sum_{|\alpha|=k+1-m} h_e^{\alpha p} |D^\alpha v; W^{m,p}(e)|^p \tag{3.8}$$

holds, if $d = 2$ or $m < k$ or $p > 2$.

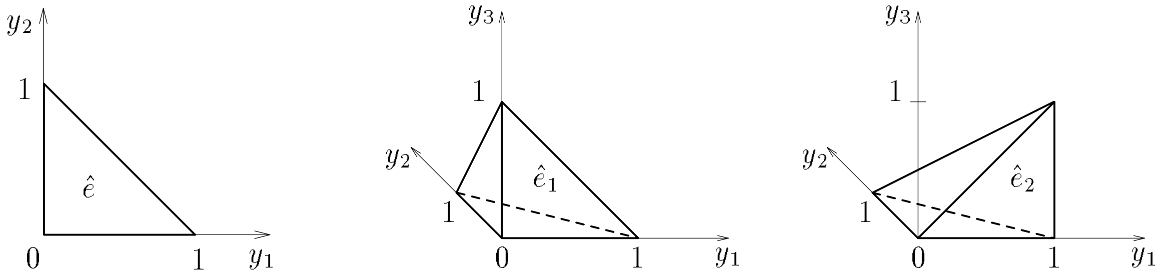

Figure 3.3: Reference elements.

Proof From Lemma 3.2 we obtain the relations

$$\left|\frac{\partial v}{\partial x_{i,e}}\right| \leq C\sum_{j=1}^{d}\min\{h_{j,e}^{-1};\ h_{i,e}^{-1}\}\left|\frac{\partial \hat{v}}{\partial y_j}\right|, \qquad \left|\frac{\partial \hat{v}}{\partial y_i}\right| \leq C\sum_{j=1}^{d}\min\{h_{j,e};\ h_{i,e}\}\left|\frac{\partial v}{\partial x_{j,e}}\right|,$$

and conclude (in multi-index notation)

$$|D^\gamma v| \leq C\sum_{|\beta|=|\gamma|} h_e^{-\beta}|D^\beta \hat{v}|, \quad |D^\beta \hat{v}| \leq C h_e^\beta \sum_{|t|=|\beta|}|D^t v|, \quad |D^\alpha \hat{v}| \leq C\sum_{|s|=|\alpha|} h_e^s |D^s v|.$$

These estimates and Theorem 3.1 imply

$$\begin{aligned}
\|D^\gamma(v - I_h^{(k)}v); L^p(e)\|^p &\leq C\,\mathrm{meas}(e)\sum_{|\beta|=|\gamma|} h_e^{-\beta p}\|D^\beta(\hat{v} - I^{(k)}\hat{v}); L^p(\hat{e})\|^p \\
&\leq C\,\mathrm{meas}(e)\sum_{|\alpha|=k+1-|\gamma|}\ \sum_{|\beta|=|\gamma|} h_e^{-\beta p}\|D^{\alpha+\beta}\hat{v}; L^p(\hat{e})\|^p \\
&\leq C\sum_{|\alpha|=k+1-|\gamma|}\ \sum_{|\beta|=|\gamma|} h_e^{-\beta p}\sum_{|t|=|\beta|}\ \sum_{|s|=|\alpha|} h_e^{\beta p}h_e^{sp}\|D^{s+t}v; L^p(e)\|^p \\
&= C\sum_{|t|=|\gamma|}\ \sum_{|s|=k+1-|\gamma|} h_e^{sp}\|D^{s+t}v; L^p(e)\|^p,
\end{aligned}$$

and the theorem can be concluded by a summation over γ, $|\gamma| = m$. □

3.4 Remarks

In this subsection, we shall shortly discuss the previous results in order to deepen the understanding of anisotropy.

Remark 3.1 Similar to [16, Theorem 2.3] we can derive a weaker anisotropic interpolation error estimate for the cases which are excluded in Theorem 3.3:

Assume that the element e fulfills the maximal angle condition and the coordinate system condition. Then for $v \in W^{k+2,p}(e)$, $I_h^{(k)}v \in \mathcal{P}_k(e)$ and $m = 0, \ldots, k$, the estimate

$$|v - I_h^{(k)}v; W^{m,p}(e)|^p \leq C\sum_{k+1-m\leq|\alpha|\leq k+2-m} h_e^{\alpha p}|D^\alpha v; W^{m,p}(e)|^p \tag{3.9}$$

holds for $d = 2, 3$ and any $p \geq 1$.

Remark 3.2 From our anisotropic error estimates we can easily derive estimates of the Jamet type by using $h_{3,e} \leq h_{2,e} \leq h_{1,e}$:

Assume that the element e satisfies the maximal angle condition. Then for $v \in W^{k+1,p}(e)$, $I_h^{(k)} v \in \mathcal{P}_k(e)$ and $m = 0, \ldots, k$, the estimate

$$|v - I_h^{(k)} v; W^{m,p}(e)| \le C h_{1,e}^{k+1-m} |v; W^{k+1,p}(e)| \tag{3.10}$$

holds, if $d = 2$ or $m < k$ or $p > 2$. If $v \in W^{k+2,p}(e)$ there holds

$$|v - I_h^{(k)} v; W^{m,p}(e)| \le C \sum_{\ell=k+1}^{k+2} h_{1,e}^{\ell-m} |v; W^{\ell,p}(e)| \tag{3.11}$$

for $d = 2, 3$, $m = 0, \ldots, k$, and any $p \ge 1$.

If we assumed the coordinate system condition the assertion follows immediately. Because the seminorms remain equivalent during a rotation of the coordinate system, the coordinate system condition can be omitted.

We remark that partial cases of this statement were proved in [5, 16, 17, 18, 24] without knowing the anisotropic estimates. We point out in particular, that the assumptions here are weaker than those in [16].

Remark 3.3 If the maximal angle condition is not fulfilled, then Theorem 3.3 is not valid. To see this, consider in the two-dimensional case the triangle with the vertices $(0,0)$, $(h_{1,e}, 0)$, $(\frac{1}{2}h_{1,e}, h_{2,e})$, and compute both sides of the estimate for $v = x_1^2$. This case leads immediately to the necessity of the maximal angle condition for the angles of the faces of a tetrahedron. Finally, an example where this condition is satisfied, but not the condition on the angles at the edges, is the tetrahedron with the vertices $(0,0,0)$, $(h,0,0)$, $(0,h,0)$, and $(\frac{1}{3}h, \frac{1}{3}h, h^\alpha)$ $(\alpha > 1)$ together with the function $v = x_1^2$. — For a discussion of the case $p = \infty$ see also [18, Examples 8, 9].

Note that in the example above the maximal angle condition related to the triangular faces is satisfied, but not for the angles at the edge. Also the converse can be true, see [18, Example 9]. That means, both conditions are independent.

Remark 3.4 An uncontrolable growth of the interpolation error for degenerate elements gives no information about the approximation error of the corresponding finite element method. In the literature one can find two examples where triangles with large angles are considered and the interpolation error in the $W^{1,2}$-norm grows to infinity. But while in [5] the finite element error grows to infinity as well, there is an example in [1] where a modified interpolate and thus the finite element solution converge.

Remark 3.5 The coordinate system condition means a suitable alignment of the mesh. Though we have seen in Remark 3.4 that a condition which is necessary for a successful interpolation may not be necessary for a good finite element approximation, we find in computations that the Galerkin/Least-squares method looses stability if the mesh is not aligned sufficiently well. The example was a convection diffusion equation in a square, with boundary conditions which produced an internal layer. Therefore the coordinate system condition should be treated carefully.

Remark 3.6 One can easily prove an anisotropic version of the inverse inequality: For $v \in \mathcal{P}_k(e)$, $k \in \mathbb{N}$ arbitrary, and $p \in [1, \infty]$, the estimate

$$\left\| \frac{\partial v}{\partial x_i}; L^p(e) \right\| \le C h_{i,e}^{-1} \|v; L^p(e)\|, \quad i = 1, \ldots, d, \tag{3.12}$$

holds if and only if the coordinate system condition is satisfied for the element e. The maximal angle condition is not necessary.

From this we can conclude

$$\|\Delta v; L^p(e)\| \leq C \left(\sum_{i=1}^{d} h_{i,e}^{-p} \left\| \frac{\partial v}{\partial x_i}; L^p(e) \right\|^p \right)^{1/p} \tag{3.13}$$

which is a slight improvement of the classical result

$$\|\Delta v; L^p(e)\| \leq C_s h_{d,e}^{-1} |v; W^{1,p}(e)| \tag{3.14}$$

that holds without the coordinate system condition. Note that $C_s = 0$ if $k = 1$.

4 A Stabilized Galerkin method

We are now prepared to treat also second derivatives of the interpolation error on anisotropic meshes. So we can consider the following Stabilized Galerkin method (also called unusual Galerkin/Least-squares method):

$$\text{Find } U_h \in V_h, \text{ such that } B_{SG}(U_h, v_h) = L_{SG}(v_h) \quad \forall v_h \in V_h \tag{SG}$$

with

$$\begin{aligned} B_{SG}(u,v) &\equiv B_G(u,v) - \sum_e \delta_e (L_\varepsilon u, L_\varepsilon v)_e, \\ L_{SG}(v) &\equiv L_G(v) - \sum_e \delta_e (f, L_\varepsilon v)_e, \end{aligned}$$

and a set $\{\delta_e\}$ of non-negative numerical diffusion parameters to be determined below. We restrict our consideration (for simplicity only) to the case $c(x) \equiv \gamma > 0$. Method (SG) with $k = 1$ and appropriately chosen δ_e is in the two dimensional case equivalent to the Galerkin scheme using piecewise linear together with piecewise cubic functions ("bubbles") [12]. The counterpart of Theorem 2.1 for this Stabilized Galerkin method reads as follows.

Theorem 4.1 *Let assumptions (H.1), (H.2) as well as*

$$\delta_e \gamma \leq \frac{1}{4}, \qquad \delta_e \varepsilon^2 C_S^2 h_{d,e}^{-2} \leq \frac{1}{4} \tag{4.1}$$

be valid with $C_S = C_S(k)$ from the inverse estimate (3.14), in particular $C_S(1) = 0$. Then there exists one and only one solution of scheme (SG) on an arbitrary admissible mesh satisfying the maximal angle condition.

Furthermore, we obtain with $\eta := u - w_h$ and arbitrary $w_h \in V_h$

$$|\!|\!| u - U_h |\!|\!|^2 \leq C \sum_e \left(\varepsilon^2 |\eta; W^{1,2}(e)|^2 + \gamma \|\eta; L^2(e)\|^2 + \varepsilon^2 h_{d,e}^2 \|\Delta \eta; L^2(e)\|^2 \right). \tag{4.2}$$

Proof (i) First of all we find from the inverse inequality (3.14) and the assumptions on the set $\{\delta_e\}$ that for $v_h \in V_h$

$$\begin{aligned} \sum_e \delta_e (L_\varepsilon v_h, L_\varepsilon v_h) &\leq 2 \sum_e \delta_e \left(\varepsilon^4 \|\Delta v_h; L^2(e)\|^2 + \gamma^2 \|v_h; L^2(e)\|^2 \right) \\ &\leq 2 \sum_e \delta_e \left(\varepsilon^4 C_S^2 h_{d,e}^{-2} |v_h; W^{1,2}(e)|^2 + \gamma^2 \|v_h; L^2(e)\|^2 \right) \\ &\leq \tfrac{1}{2} |\!|\!| v_h |\!|\!|^2. \end{aligned} \tag{4.3}$$

The V_h-ellipticity with respect to $|||\,.\,|||^2 \equiv B_G(.,.)$ follows then via

$$B_{SG}(v_h, v_h) = |||\, v_h \,|||^2 - \sum_e \delta_e (L_\varepsilon v_h, L_\varepsilon v_h) \geq \tfrac{1}{2} |||\, v_h \,|||^2. \tag{4.4}$$

The continuity of B_{SG} and L_{SG} on $V_h \times V_h$ and V_h, respectively, is concluded from (4.3) and standard inequalities:

$$\begin{aligned}
|B_{SG}(u_h, v_h)| &\leq \varepsilon^2 |u_h; W^{1,2}(e)||v_h; W^{1,2}(e)| + \gamma \|u_h; L^2(e)\| \|v_h; L^2(e)\| + \\
&\qquad + \sum_e \delta_e \|L_\varepsilon u_h; L^2(e)\| \|L_\varepsilon v_h; L^2(e)\| \\
&\leq \left(|||\, u_h \,|||^2 + \sum_e \delta_e \|L_\varepsilon u_h; L^2(e)\|^2 \right)^{1/2} \left(|||\, v_h \,|||^2 + \sum_e \delta_e \|L_\varepsilon v_h; L^2(e)\|^2 \right)^{1/2} \\
&\leq \tfrac{3}{2} |||\, u_h \,||| \, |||\, v_h \,|||, \\
|L_{SG}(v_h)| &\leq \|f; L^2(\Omega)\| \left[\|v_h; L^2(\Omega)\| + \left(\sum_e \delta_e \cdot \delta_e \|L_\varepsilon v_h; L^2(e)\|^2 \right)^{1/2} \right] \\
&\leq \|f; L^2(\Omega)\| \left[\frac{1}{\gamma} + \left(\frac{1}{4\gamma} \cdot \frac{1}{2} \right)^{1/2} \right] |||\, v_h \,|||.
\end{aligned}$$

This implies existence and uniqueness of the discrete solution U_h.

(ii) We introduce the splitting

$$e \equiv u - U_h = (u - w_h) + (w_h - U_h) \equiv \eta + \chi_h \tag{4.5}$$

with an arbitrary $w_h \in V_h$. Using (4.3), the Cauchy-Schwarz inequality, the inverse inequality and the Young inequality we derive

$$\begin{aligned}
\frac{1}{2} |||\, \chi_h \,|||^2 &\leq B_{SG}(\chi_h, \chi_h) \;=\; B_{SG}(e - \eta, \chi_h) \;=\; -B_{SG}(\eta, \chi_h) \;\leq\; |B_{SG}(\eta, \chi_h)| \\
&\leq \Big| \sum_e \Big(\gamma(1 - \delta_e \gamma)\eta, \chi_h)_e + \varepsilon^2 (\nabla \eta, \nabla \chi_h)_e + \varepsilon^2 \delta_e (\Delta \eta, \gamma \chi_h)_e + \\
&\qquad + \delta_e (-\varepsilon^2 \Delta \eta, \varepsilon^2 \Delta \chi_h)_e + \delta_e (\gamma \eta, \varepsilon^2 \Delta \chi_h)_e \Big) \Big| \\
&\leq \frac{1}{8} \gamma \|\chi_h; L^2(\Omega)\|^2 + 2 \sum_e \gamma (1 - \delta_e \gamma)^2 \|\eta; L^2(e)\|^2 \\
&\qquad + \frac{1}{8} \varepsilon^2 |\chi_h; W^{1,2}(\Omega)|^2 + 2\varepsilon^2 |\eta; W^{1,2}(\Omega)|^2 \\
&\qquad + \frac{1}{8} \gamma \|\chi_h; L^2(\Omega)\|^2 + 2 \sum_e \varepsilon^4 \delta_e^2 \gamma \|\Delta \eta; L^2(e)\|^2 \\
&\qquad + \frac{\varepsilon^2}{4} \sum_e \varepsilon^2 \delta_e C_S^2 h_{d,e}^{-2} |\chi_h; W^{1,2}(e)^d\|^2 + \sum_e \varepsilon^4 \delta_e \|\Delta \eta; L^2(e)^d\|^2 \\
&\qquad + \frac{\varepsilon^2}{4} \sum_e \varepsilon^2 \delta_e C_S^2 h_{d,e}^{-2} |\chi_h; W^{1,2}(e)|^2 + \sum_e \delta_e \gamma^2 \|\eta; L^2(e)^d\|^2.
\end{aligned}$$

This implies together with the assumptions on δ_e that

$$\frac{1}{4} |||\, \chi_h \,|||^2 \leq \sum_e \left(\frac{9}{4} \gamma \|\eta; L^2(e)\|^2 + 2\varepsilon^2 |\eta; W^{1,2}(e)|^2 + \frac{1}{4} C_S^{-2} \varepsilon^2 h_{d,e}^2 \left(2\frac{1}{4} + 1 \right) \|\Delta \eta; L^2(e)\|^2 \right)$$

and together with the triangle inequality the assertion. □

Theorem 4.1 leads in the case of an isotropic mesh via standard interpolation error estimates to

$$\varepsilon^2|u-U_h;W^{1,2}(\Omega)|^2+\gamma\|u-U_h;L^2(\Omega)\|^2\le C\sum_e h_{1,e}^{2k}\left(\varepsilon^2+\gamma h_{1,e}^2\right)|u;W^{k+1,2}(e)|^2. \quad (4.6)$$

The case $k=1$, $d=2$ was already treated in [12] with a different result, see the discussion below.

Note that δ_e can be chosen as

$$\delta_e=\delta_0\frac{h_{d,e}^2}{\varepsilon^2+\gamma h_{d,e}^2},\qquad \delta_0\in\left[0,\tfrac14\min\{1,C_S^{-2}\}\right] \quad (4.7)$$

to satisfy condition (4.1).

We consider now the case of a Shishkin type mesh. For simplicity only, we restrict ourselves to the two dimensional situation with $\Omega=(0,1)^2$.

Theorem 4.2 *Assume that the exact solution of (1.1), (1.2) with $c(x)\equiv\gamma$ in $\Omega=(0,1)^2$ satisfies the assumptions (2.8), (2.11), (2.24) and (2.26). Furthermore, consider a Shishkin type mesh constructed according to (2.32) with $a\ge a_0\frac{\varepsilon}{\gamma_0}|\log\varepsilon|$, $a_0\ge k+\frac12$ and set δ_e as in (4.7). Then the error of method (SG) satisfies*

$$\varepsilon^2|u-U_h;W^{1,2}(\Omega)|^2+\gamma\|u-U_h;L^2(\Omega)\|^2\le Ch^{2k}\left(\varepsilon(\ln\tfrac1\varepsilon)^{2k}+h^2\right). \quad (4.8)$$

Proof We insert the local interpolation error estimates of Theorem 3.3 into estimate (4.2). The remainder of the proof is in analogy to the proof of Theorem 2.8. □

The result is the same as for the Galerkin method but we have some freedom in the choice of the set $\{\delta_e\}$ to minimize the error. To examine the influence of this choice we calculated the test example of Section 2.4 with $a=2\varepsilon\log_{10}\frac1\varepsilon$ and $\delta_0=0.25,\ 0.125,\ 0.0625$, and for comparison $\delta_0=0.125$. The results show in few cases with $\delta_0=-0.0625$ an improvement in comparison with the error of the pure Galerkin method, however only by less than 1%. The error is slightly increasing for higher and negative values of δ_0. This shows that the resolution of the layer with anisotropic meshes makes a stabilization superfluous. One can work with a pure Galerkin method.

Finally, we will prove another error estimate for the special case $k=1$. Here, the consideration is simplified because the Laplacian of functions from the approximating space vanishes. In the next theorem we will repeat a result of Franca and Farhat [12] and extend it to anisotropic meshes. In contrast to that paper we will formulate the proof in a way that motivates the choice of the parameters δ_e.

Theorem 4.3 *In the isotropic case the error of the Stabilized Galerkin method (SG) satisfies for*

$$\delta_e=\frac{h_{d,e}^2}{\varepsilon^2+\gamma h_{d,e}^2} \quad (4.9)$$

the estimate

$$\varepsilon^2|u-U_h;W^{1,2}(\Omega)|^2+\sum_e\frac{\gamma\varepsilon^2}{\varepsilon^2+\gamma h_{d,e}^2}\|u-U_h;L^2(e)\|^2\le C\varepsilon^2h_{1,e}^2|u;W^{2,2}(\Omega)|^2. \quad (4.10)$$

Under the assumptions on the solution u and on the family of meshes $\mathcal{T}_h$ as in Theorem 4.2, and using δ_e from (4.9) the error estimate is improved to

$$\varepsilon^2|u-U_h;W^{1,2}(\Omega)|^2+\sum_e\frac{\gamma\varepsilon^2}{\varepsilon^2+\gamma h_{d,e}^2}\|u-U_h;L^2(e)\|^2\le C\varepsilon h^2\left(\ln\tfrac1\varepsilon\right)^2. \quad (4.11)$$

Proof We prove the second estimate because the first one was already proved in [12]. The first step is to transform the estimation of the finite element error to a general approximation problem using some ideas from part (ii) of the proof of Theorem 4.1. But modifications are necessary because our final choice (4.9) for δ_e satisfies the assumption (4.1) only in the asymptotic range which is not interesting for practical calculations. That means we can not use (4.4) in the error analysis. We derive only

$$B_{SG}(\chi_h, \chi_h) \leq 2\sum_e \left(\gamma(1-\delta_e\gamma)\|\eta; L^2(e)\|^2 + \frac{1}{2}\varepsilon^2|\eta; W^{1,2}(e)|^2 + \frac{\varepsilon^4\delta_e^2\gamma}{1-\delta_e\gamma}\|\Delta\eta; L^2(e)\|^2 \right).$$

This leads to the somewhat weaker L^2-part in the error estimates (4.10) and (4.11) in comparison to (4.2) and (4.8): Using (4.5) we obtain

$$\begin{aligned} &\varepsilon^2|u-U_h; W^{1,2}(\Omega)|^2 + \sum_e \gamma(1-\delta_e\gamma)\|u-U_h; L^2(e)\|^2 \\ &\leq \; C\sum_e \left(\gamma(1-\delta_e\gamma)\|\eta; L^2(e)\|^2 + \varepsilon^2|\eta; W^{1,2}(e)|^2 + \frac{\varepsilon^4\delta_e^2\gamma}{1-\delta_e\gamma}\|\Delta\eta; L^2(e)\|^2 \right) \end{aligned}$$

Inserting $\eta = u - I_h^{(1)}u$ leads via Theorem 3.3 to the estimate

$$\begin{aligned} &\varepsilon^2|u-U_h; W^{1,2}(\Omega)|^2 + \sum_e \gamma(1-\delta_e\gamma)\|u-U_h; L^2(e)\|^2 \\ &\leq \; C\sum_e \sum_{|\alpha|=1} \sum_{|\beta|=1} \left(\gamma(1-\delta_e\gamma)h_e^{2(\alpha+\beta)} + \varepsilon^2 h_e^{2\alpha} + \frac{\varepsilon^4\delta_e^2\gamma}{1-\delta_e\gamma} \right) \|D^{\alpha+\beta}u; L^2(e)\|^2. \end{aligned}$$

By considering the different cases Ω_i as in the proofs in Subsection 2.3 and using the assumptions on u, we obtain the following upper bound.

$$\begin{aligned} &\varepsilon^2|u-U_h; W^{1,2}(\Omega)|^2 + \sum_e \gamma(1-\delta_e\gamma)\|u-U_h; L^2(e)\|^2 \\ &\leq \; C\varepsilon h^2 \left(\ln\tfrac{1}{\varepsilon}\right)^2 \max_e \left(\gamma(1-\delta_e\gamma)\varepsilon^{-2}h_{d,e}^2 + 1 + \frac{\varepsilon^2\delta_e^2\gamma}{1-\delta_e\gamma}h_{d,e}^{-2} \right). \end{aligned}$$

Analyzing the expression in parentheses we find that just the choice (4.9) for δ_e leads to a upper bound which is bounded by a constant (independent of h and ε). So we conclude the assertion (4.11).

Finally we remark that this choice of δ_e guarantees that $(B_{SG}(.,.))^{1/2}$ is a norm in V_h and, consequently, the existence and uniqueness of the discrete solution.

Using the ideas of this proof for $k \geq 2$ leads to a different choice of δ_e, for which $B_{SG}(.,.)$ does not define a scalar product in V_h. This is the reason why Theorem 4.3 is restricted to $k = 1$. □

In comparison to Theorems 4.1 and 4.2 we point out that the error estimate in Theorem 4.3 is better with respect to the $W^{1,2}$-seminorm, but weaker with respect to the L^2-norm. But asymptotically for $h \to 0$ the factor in front of the L^2-part tends also to γ, as in (4.6) and (4.8).

5 More general domains

We conjecture that the ideas of anisotropic mesh refinement can be used for more general polygonal/polyhedral domains as well, for the following reasons.

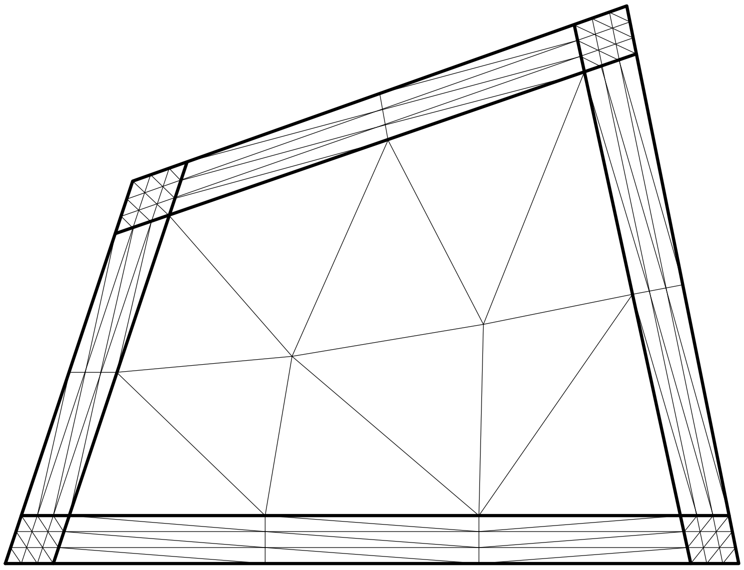

Figure 5.1: Anisotropic mesh in the boundary layer of a general polygonal domain.

First, the idea of the domain decomposition can be generalized in a straightforward manner, see Figure 5.1 for an illustration.

Second, the resulting domains are no longer rectangles or cubes. Therefore the resulting meshes are not of tensor product type as before. However, we did never use this property in the proofs for our error estimates. As we have seen in Section 3 the anisotropic interpolation error estimates hold true for more general simplicial elements which satisfy the maximal angle condition and the coordinate system condition, here with respect to boundary fitted coordinates as introduced in Subsection 2.2.

Third, the critical point is that the pointwise estimates of the partial derivatives of the exact solution were given in [22] only for domains $(0,1)^d$, $d = 2,3$. However, we suppose that the principal properties of the solution carry over to more general domains.

Singularities of r^λ-type in the neighbourhood of the non-smooth part of the boundary are a different matter. If they are not avoided by compatibility conditions they must be treated by a superposition of another mesh refinement. The theory is well developed for diffusion dominated problems, see for example [1, 4] and the references therein.

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References

[1] Th. Apel and M. Dobrowolski. Anisotropic interpolation with applications to the finite element method. *Computing*, 47:277–293, 1992.

[2] Th. Apel and G. Lube. Local inequalities for anisotropic finite elements and their application to convection-diffusion problems. Preprint SPC94_26, TU Chemnitz-Zwickau, 1994.

[3] Th. Apel and G. Lube. Anisotropic mesh refinement in stabilized Galerkin methods. *Numer. Math.*, 74:261–282, 1996.

[4] Th. Apel, A.-M. Sändig, and J. R. Whiteman. Graded mesh refinement and error estimates for finite element solutions of elliptic boundary value problems in non-smooth domains. *Math. Meth. Appl. Sci.*, 19:63–85, 1996.

[5] I. Babuška and A. K. Aziz. On the angle condition in the finite element method. *SIAM J. Numer. Anal.*, 13:214–226, 1976.

[6] N. S. Bakhvalov. Optimization of methods for the solution of boundary value problems in the presence a of boundary layer. *Zh. Vychisl. Mat. Mat. Fiz.*, 9:841–859, 1969. Russian.

[7] I. A. Blatov. The Galerkin finite-element method for elliptic quasilinear singularly perturbed boundary value problems. I. *Differ. Uravn.*, 28:1168–1177, 1992. Russian.

[8] I. A. Blatov. The Galerkin finite-element method for elliptic quasilinear singularly perturbed boundary value problems. II. *Differ. Uravn.*, 28:1799–1810, 1992. Russian.

[9] I. A. Blatov. The Galerkin finite-element method for elliptic quasilinear singularly perturbed boundary value problems. III: Problems with corner boundary layers. *Differ. Uravn.*, 30:467–479, 1994. Russian.

[10] V. F. Butuzov. The asymptotic properties of solutions of the equation $\mu^2\Delta u - \kappa^2(x,y)u = f(x,y)$ in a rectangle. *Differencialnye Uravneniya*, 9:1274–1279, 1973.

[11] P. Ciarlet. *The finite element method for elliptic problems.* North-Holland, Amsterdam, 1978.

[12] L. P. Franca and C. Farhat. Bubble functions prompt unusual stabilized finite element methods. *Comp. Meth. Appl. Mech. Engrg.*, 123:299–308, 1995.

[13] H. Goering, A. Felgenhauer, G. Lube, H.-G. Roos, and L. Tobiska. *Singularly perturbed differential equations.* Akademie-Verlag, Berlin, 1983.

[14] H. Han and R. B. Kellogg. Differentiability properties of solutions of the equation $-\varepsilon^2\Delta u + ru = f(x,y)$ in a square. *SIAM J. Math. Anal.*, 21:394–408, 1990.

[15] F. A. Howes. Singularly perturbed semilinear elliptic boundary value problems. *Comm. Part. Diff. Equations*, 4:1–39, 1979.

[16] P. Jamet. Estimations d'erreur pour des éléments finis droits presque dégénérés. *R.A.I.R.O. Anal. Numér.*, 10:43–61, 1976.

[17] M. Křížek. On semiregular families of triangulations and linear interpolation. *Appl. Math.*, 36:223–232, 1991.

[18] M. Křížek. On the maximum angle condition for linear tetrahedral elements. *SIAM J. Numer. Anal.*, 29:513–520, 1992.

[19] J. J. H. Miller, E. O'Riordan, and G. I. Shishkin. *Fitted numerical methods for singular perturbation problems — Error estimates in the maximum norm for linear problems in one and two dimensions.* World Scientific Publishing Co., Singapore, 1996.

[20] A. H. Schatz and L. B. Wahlbin. On the finite element method for singularly perturbed reaction–diffusion problems in two and one dimensions. *Math. Comp.*, 40:47–89, 1983.

[21] G. I. Shishkin. Approximation of solutions of singularly perturbed boundary value problems with a corner boundary layer. *Zh. Vychisl. Mat. Mat. Fiz.*, 27:1360–1374, 1987.

[22] G. I. Shishkin. *Mesh approximations of singularly perturbed elliptic and parabolic problems.* Russ. Acad. Sci., Ekaterinburg, 1992. (Russian).

[23] M. Stynes and E. O'Riordan. A uniformly convergent Galerkin method on a Shishkin mesh for a convection-diffusion problem. Preprint 1996 – 6, Department of Mathematics, University College Cork (Ireland),, 1996.

[24] J. L. Synge. *The hypercircle in mathematical physics.* Cambridge University Press, Cambridge, 1957.

[25] Chr. Xenophontos. *The hp finite element method for singularly perturbed problems.* PhD thesis, University of Maryland, 1996.

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