Technische Universität Chemnitz

Sonderforschungsbereich 393

Numerische Simulation auf massiv parallelen Rechnern

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Clément-type interpolation on spherical domains – interpolation error estimates and application to a posteriori error estimation

Preprint SFB393/03-13

Abstract In this paper, a mixed boundary value problem for the Laplace-Beltrami operator is considered for spherical domains in $\mathbb{R}^3$, i.e. for domains on the unit sphere. These domains are parametrized by spherical coordinates (φ, θ), such that functions on the unit sphere are considered as functions in these coordinates. Careful investigation leads to the introduction of a proper finite element space corresponding to an isotropic triangulation of the underlying domain on the unit sphere. Error estimates are proven for a Clément-type interpolation operator, where appropriate, weighted norms are used. The estimates are applied to the deduction of a reliable and efficient residual error estimator for the Laplace-Beltrami operator.

Key Words Clément-type interpolation, spherical domains, Laplace-Beltrami operator, a posteriori error estimation

AMS subject classification 65N15; 65N30, 65N50

Preprint-Reihe des Chemnitzer SFB 393

ISSN 1619-7178 (Print)   **ISSN 1619-7186** (Internet)

SFB393/03-13   **July 2003**

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1 Introduction

The finite element method is suited to solve partial differential equations approximately. Nowadays one is not only interested in asymptotic a priori estimates of the discretization error but also in reliable and efficient a posteriori error estimates based on a finite element solution on a given mesh, see for example the monographs by Verfürth [17] and Ainsworth/Oden [1] for an overview.

The boundary value problems treated in the literature on a posteriori error estimation are posed on two- or three-dimensional domains. In our paper, however, we focus on the mixed boundary value problem for the Laplace-Beltrami equation over a domain that is part of the unit sphere $\mathcal{S}^2 \subset I\!R^3$. Our interest in spherical domains arose from the computation of three-dimensional corner singularities for elliptic operators like the Laplace or the Lamé operator [2]. The quantitative knowledge of these singularities is important for analysts and engineers, for example for the onset of cracks, see work by Leguillon [13, 14, 15], and Andrä/Dimitrov/Schnack [7, 16]. The singularities can be computed by solving operator eigenvalue problems which are defined over spherical domains $\Omega \subset \mathcal{S}^2$.

The parametrization of a two-dimensional manifold $\Omega \subset I\!R^3$ leads to a problem in a domain $G \subset I\!R^2$ in the parameter space so that we can try to apply techniques for error estimation similar to those known for plane domains. The main difficulty is that the nature of the problem is determined by the spherical domain and that the variable transformation F with $\Omega = F(G)$ influences the operator on G and also the norms over G that have to be used for error estimation.

Generally the parametrization is not unique. A common parametrization of the unit sphere is given by spherical coordinates (φ, θ), although it possesses singularities at the poles. Treating functions which are defined on the unit sphere as functions of the two parameters φ and θ, the usual norms and gradients have to be provided with certain weights. Accordingly, the definition of other ingredients like an interpolation operator has to be adapted. We call Ω and G *regular* if G is open, connected and polygonal and if its boundary is piecewise parallel to the φ- or θ-axes. For simplicity we consider in this paper only domains (Ω or G, respectively) that are regular.

Typical for the investigation of a posteriori error estimators is the use of an interpolation operator that is able to act on functions from the Sobolev space H^1, for example the interpolation operator of Clément [6], see papers by Carstensen/Funken [5] and Verfürth [18]. One of the aims of this paper is to modify the operator appropriately for the use on spherical domains and to prove the necessary estimates for the interpolation error.

In Sections 2 the model problem is introduced. We consider a mixed boundary value problem for the Laplace-Beltrami operator. This operator finds application, for instance, in conductivity or heat transfer problems in curved surfaces. The finite element discretization is discussed in Section 3. We focus in particular on the definition of a finite element mesh over Ω which is the image of a mesh over G (with straight edges) and consists of shape-regular (isotropic) elements on the spherical domain Ω. Due to the singularity of the transformation, one cannot achieve meshes that are shape-regular both over G and Ω. After stating the trace theorem and a Bramble-Hilbert type lemma in the form suitable for our application (the proof is very technical and postponed to the Appendix), we prove in Section 4 an error estimate for a weighted L^2-projection. These basic estimates allow then to investigate a Clément-type interpolation operator in Section 5. As pointed out above, error estimates for this interpolation operator are an essential ingredient for the proof of bounds of a residual a posteriori error estimator. We demonstrate this for our model problem in Section 6. The extension to the eigenvalue problems mentioned above is a topic of future research.

Throughout this paper, constants do not depend on the triangulation of the computational domain nor on the functions under consideration. Instead of $\psi_1 \leq c\psi_2$ and $c_1\psi_1 \leq \psi_2 \leq c_2\psi_1$ with certain constants c, c_1, c_2, we will write $\psi_1 \lesssim \psi_2$ and $\psi_1 \sim \psi_2$, respectively.

2 The model problem

2.1 The mixed problem for the Laplace-Beltrami operator

We consider an open, connected subset Ω of the unit sphere $\mathcal{S}^2$. Its boundary $\Gamma = \partial\Omega$ is split into *Dirichlet boundary* Γ_D and *Neumann boundary* Γ_N, such that $\Gamma = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$, and such that Γ_D is closed with respect to Γ.

Denote by $\Delta_{\mathcal{S}}$ the Laplace operator on $\mathcal{S}^2$, the so called *Laplace-Beltrami operator*, and consider the following model problem:

$$\begin{array}{rcll} -\Delta_{\mathcal{S}} u &=& f & \text{in } \Omega \\ u &=& 0 & \text{on } \Gamma_D \\ \frac{\partial u}{\partial \mathbf{n}} &=& g & \text{on } \Gamma_N, \end{array} \tag{1}$$

where $\mathbf{n}$ is the exterior normal to Ω, i.e., $\mathbf{n}(\mathbf{x})$ lies in the tangential plane at $\mathbf{x} \in \partial\Omega$ and is orthogonal to the tangential vector at $\mathbf{x}$. In order to ensure the existence of a unique solution, we assume that Γ_D has positive length.

Let $\nabla_{\mathcal{S}}$ be the *spherical gradient* of a function. With the symbols $\|\lceil \cdot \rceil\|_{k,S}$ and $\lceil \cdot \rceil_{k,S}$, we denote the *Sobolev norms* and *seminorms* of order k, $k = 0, 1$, for any subset $S \subset \Omega$, i.e.

$$\begin{aligned} \lceil u \rceil_{0,S} = \|\lceil u \rceil\|_{0,S} &:= \left(\int\limits_S |u|^2 \,\mathrm{d}\omega \right)^{1/2}, \\ \lceil u \rceil_{1,S} &:= \left(\int\limits_S |\nabla_{\mathcal{S}} u|^2 \,\mathrm{d}\omega \right)^{1/2}, \end{aligned}$$

where $\mathrm{d}\omega$ is the surface element and $\|\lceil u \rceil\|_{1,S}^2 = \lceil u \rceil_{0,S}^2 + \lceil u \rceil_{1,S}^2$. Correspondingly, let $\mathrm{d}\sigma$ be the line element.

By analogy to the usual Sobolev spaces, we introduce *Sobolev spaces over spherical domains*, denoted by $\mathcal{H}^k(S)$, which consist of functions u with bounded norms $\|\lceil u \rceil\|_{\ell,S}$, $0 \leq \ell \leq k$, $k = 0, 1$. Similarly, norms and Sobolev spaces of higher order can be defined; see for example Kozlov, Maz'ya, Roßmann [9].

Define the space $X = \{v \in \mathcal{H}^1(\Omega) \mid v = 0 \text{ on } \Gamma_D\}$. Using Green's formula, Problem (1) can be transferred into its weak formulation: Find $u \in X$ such that

$$\int\limits_\Omega \nabla_{\mathcal{S}} u \cdot \nabla_{\mathcal{S}} v \,\mathrm{d}\omega = \int\limits_\Omega f v \,\mathrm{d}\omega + \int\limits_{\Gamma_N} g v \,\mathrm{d}\sigma \quad \forall v \in X. \tag{2}$$

2.2 Parametrization: Spherical coordinates

For our further calculations it is useful to parametrize the unit sphere. To this end, we choose spherical coordinates and consider the transformation

$$x = \cos\varphi \sin\theta, \quad y = \sin\varphi \sin\theta, \quad z = \cos\theta$$

with $\varphi \in [0, 2\pi)$, $\theta \in [0, \pi]$, where x, y, z denote the Cartesian coordinates. Then, the Laplace-Beltrami operator explicitly reads

$$\Delta_{\mathcal{S}} u = \frac{1}{\sin^2\theta} \frac{\partial^2 u}{\partial \varphi^2} + \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \Big(\sin\theta \frac{\partial u}{\partial\theta} \Big).$$

The representation of the gradient in spherical coordinates is given by

$$\nabla_{\mathcal{S}} u = \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\theta} \right)^{\top}.$$

All functions considered in the following shall be expressed by the two parameters φ and θ. The parameter domain (in the (φ, θ)-coordinate system) corresponding to Ω will be denoted by G, i.e. $G \subset [0, 2\pi) \times [0, \pi]$.

Curves γ on the unit sphere shall always be given in parametrized form $\gamma = \gamma(t) = \{(\varphi(t), \theta(t)) \mid t \in [0, 1]\}$. The corresponding line and surface elements read

$$\mathrm{d}\sigma = \sqrt{\dot\varphi(t)^2 \sin^2\theta(t) + \dot\theta(t)^2}\, \mathrm{dt} \qquad \text{and} \qquad \mathrm{d}\omega = \sin\theta \,\mathrm{d}\varphi \,\mathrm{d}\theta,$$

respectively. If we insert these explicit forms into the weak formulation (2), we actually have to use G and the parametrized counterpart of Γ_N as the integration domains for the surface integrals and the boundary integral, respectively. In order to keep the amount of notation at a minimum, we will carry on writing Γ_D and Γ_N either meaning Dirichlet and Neumann boundary of Ω or their parametrized forms in the parameter domain G. Their concrete meanings are always clear from the context. Note that, in general, ∂G is *not* the parametrization of $\partial\Omega$, compare for example $\Omega = \mathcal{S}^2, \partial\Omega = \emptyset$, but ∂G is the boundary of the rectangle $[0, 2\pi] \times [0, \pi]$.

Hence, using the explicit forms of $\mathrm{d}\sigma$ and $\mathrm{d}\omega$, Formulation (2) equals: Find $u \in X$ such that

$$\int\limits_G \nabla_{\mathcal{S}} u \cdot \nabla_{\mathcal{S}} v \sin\theta \,\mathrm{d}\varphi \,\mathrm{d}\theta = \int\limits_G f v \sin\theta \,\mathrm{d}\varphi \,\mathrm{d}\theta + \int\limits_{\Gamma_N} g v \,\mathrm{d}\sigma \quad \forall v \in X. \tag{3}$$

We remark that the symbols $|\cdot|_{k,S}$, $\|\cdot\|_{k,S}$ and $\|\cdot\|_{0,\gamma}$ denote the usual Sobolev seminorms and norms for plane domains S or curves $\gamma \subset I\!R^2$, i.e.

$$\begin{aligned}
|u|_{0,S} &= \Big(\int\limits_S |u|^2 \,\mathrm{d}\varphi \,\mathrm{d}\theta \Big)^{1/2}, \\
|u|_{1,S} &= \Big(\int\limits_S |\nabla u|^2 \,\mathrm{d}\varphi \,\mathrm{d}\theta \Big)^{1/2} = \Big(\int\limits_S |\partial u/\partial\varphi|^2 + |\partial u/\partial\theta|^2 \,\mathrm{d}\varphi \,\mathrm{d}\theta \Big)^{1/2}, \\
\|u\|_{0,\gamma} &= \Big(\int\limits_0^1 |u(\varphi(t), \theta(t))|^2 \sqrt{\dot\varphi(t)^2 + \dot\theta(t)^2} \,\mathrm{dt} \Big)^{1/2}.
\end{aligned}$$

The symbols $|S| := \|1\|_{0,S}^2$ and $|\gamma| := \|1\|_{0,\gamma}^2$ are used to express the area of a domain $S \subset I\!R^2$ and the length of a curve $\gamma \subset I\!R^2$, respectively. Analogously, the spherical size or length of a domain $S \subset \mathcal{S}^2$ or a curve $\gamma \subset \mathcal{S}^2$ are denoted by $\lceil S \rceil := \lVert\!\lceil 1 \rceil\!\rVert_{0,S}^2$ and $\lceil \gamma \rceil := \lVert\!\lceil 1 \rceil\!\rVert_{0,\gamma}^2$.

3 Finite element discretization

For the numerical solution of partial differential equations, it is necessary to discretize the given problem. In particular, we consider a family $\mathcal{T}_h$, $h > 0$, of triangulations of Ω into elements $T \subset \mathcal{S}^2$ with standard assumptions, i.e.

(i) $\overline{\Omega} = \bigcup_{T \in \mathcal{T}_h} \overline{T}$, $T_i \cap T_j = \emptyset$ for $i \neq j$;

(ii) the closures of any two elements $T_i \neq T_j$ of $\mathcal{T}_h$ are either disjoint or have exactly one common edge or one common vertex;

(iii) the number of elements which have a common vertex is bounded from above by a fixed constant $\mathcal{Z}$ independent of h and $\mathcal{T}_h$.

Without loss of generality, we assume that all triangles and edges are open. We denote the sets of all edges and vertices in the triangulation $\mathcal{T}_h$ by $\mathcal{E}_h$ and $\mathcal{N}_h$, respectively. The edges and vertices of an element $T \in \mathcal{T}_h$ are denoted by $\mathcal{E}(T)$ and $\mathcal{N}(T)$. Analogously, let $\mathcal{N}(E)$ be the set of the two vertices of an edge $E \in \mathcal{E}_h$. According to the decomposition of Γ, we define

$$\begin{aligned}
\mathcal{E}_{h,D} &:= \{E \in \mathcal{E}_h \mid E \subset \Gamma_D\}, & \mathcal{N}_{h,D} &:= \mathcal{N}_h \cap \Gamma_D, \\
\mathcal{E}_{h,N} &:= \{E \in \mathcal{E}_h \mid E \subset \Gamma_N\}, & \mathcal{N}_{h,N} &:= \mathcal{N}_h \cap \Gamma_N, \\
\mathcal{E}_{h,\Omega} &:= \mathcal{E}_h \setminus (\mathcal{E}_{h,D} \cup \mathcal{E}_{h,N}), & \mathcal{N}_{h,\Omega} &:= \mathcal{N}_h \setminus (\mathcal{N}_{h,D} \cup \mathcal{N}_{h,N}).
\end{aligned}$$

For $T \in \mathcal{T}_h$, $E \in \mathcal{E}_h$ and $x \in \mathcal{N}_h$, we introduce

$$\begin{aligned}
\omega_T &:= \bigcup_{\mathcal{E}(T) \cap \mathcal{E}(T') \neq \emptyset} T', & \omega_E &:= \bigcup_{E \in \mathcal{E}(T')} T', & \omega_x &:= \bigcup_{x \in \mathcal{N}(T')} T', \\
\tilde{\omega}_T &:= \bigcup_{\mathcal{N}(T) \cap \mathcal{N}(T') \neq \emptyset} T', & \tilde{\omega}_E &:= \bigcup_{\mathcal{N}(E) \cap \mathcal{N}(T') \neq \emptyset} T'.
\end{aligned}$$

see Figure 1.

There are two options for the choice of a proper triangulation. Either the parameter domain G is meshed isotropically, i.e., each element has approximately the same plane dimensions in φ-direction as in θ-direction; or the sphere is meshed isotropically, i.e., the spherical dimensions in both directions are equivalent for each triangle. The Figures 2 and 3 show samples for both kinds of triangulations in the parameter domain and their counterparts on the sphere.

The isotropic triangulation of the parameter domain is easier to implement. On the other hand, it yields anisotropic elements on the sphere. Near the poles, elements narrow more and more in φ-direction. Careful analysis of a residual error estimator reveals that the upper bound for the error depends on the maximum of the two dimensions of each element while the lower bound depends on their minimum, which may result in a reliability-efficience gap in the order of their aspect ratio. Therefore, we will generally get a reliable

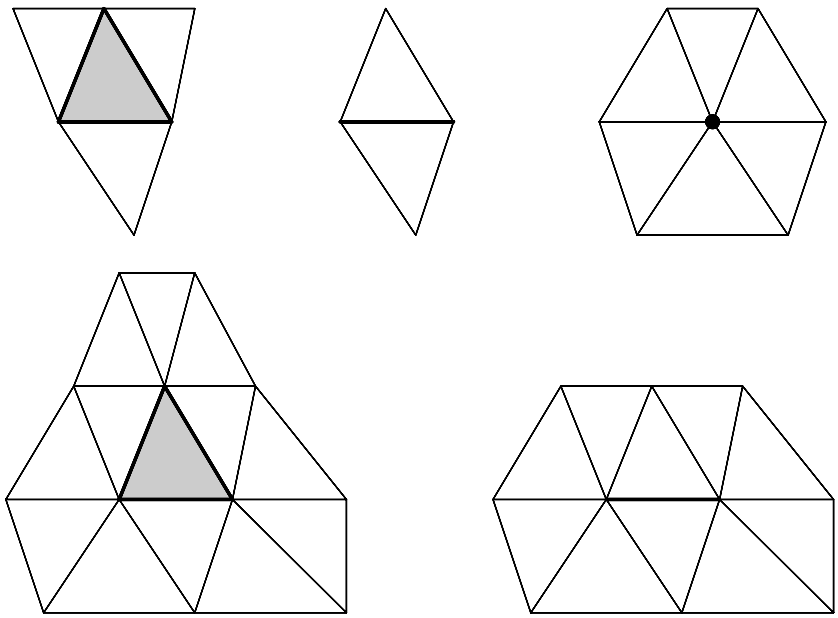


Figure 1: The domains ω_T, ω_E, ω_x, $\tilde{\omega}_T$, $\tilde{\omega}_E$ (from left top to right bottom)

and efficient error estimator only if this maximum and this minimum are equivalent. This equivalence is requested in Condition (5) on the mesh.

We remark that similar considerations occur in the analysis of error estimators for anisotropic discretizations, as carried out, for example, by Kunert [11, 12]. There, the discrepancy is remedied by the introduction of an alignment measure (matching function) which has small values, if the mesh corresponds to some anisotropic function. In our application, however, the anisotropic mesh is purely artificial and the alignment measure may produce values in the order of the aspect ratio of the elements on the spherical domain.

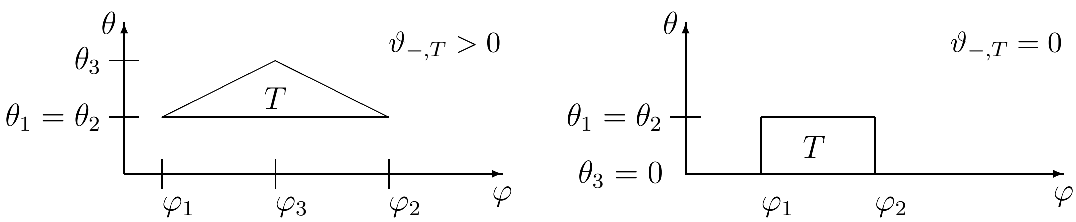
This is why we consider a discretization of the spherical domain, where both dimensions of any element are approximately the same, i.e., they only differ by a constant which is independent of the triangulation. To this end, it is necessary to use an isotropic triangulation of the sphere as displayed in Figure 3. However, the elements in the parameter domain become anisotropic then. Therefore, we make the following assumptions on the mesh:

Conformity The conditions (i)–(iii) hold.

Axiparallel triangles The nodes $\underline{a}_{i,T} = (\varphi_{i,T}, \theta_{i,T})$, $i = 1, 2, 3$, of an element $T \in \mathcal{T}_h$ satisfy

$$\varphi_{1,T} \leq \varphi_{3,T} \leq \varphi_{2,T} \quad \text{and} \quad \theta_{1,T} = \theta_{2,T}, \tag{4}$$

see also the following picture.



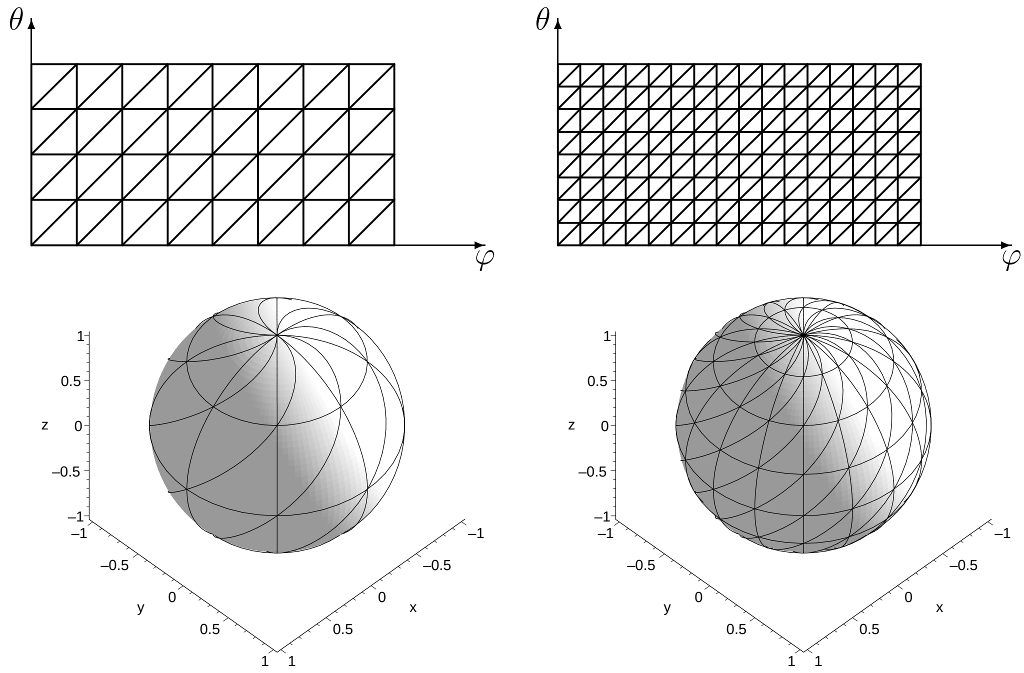


Figure 2: Isotropic triangulation of the parameter domain (top) with $n_\theta \in \{4, 8\}$ divisions in θ-direction and $n_\varphi = 2n_\theta$ divisions in φ-direction and the corresponding anisotropic triangulation on the sphere (bottom)

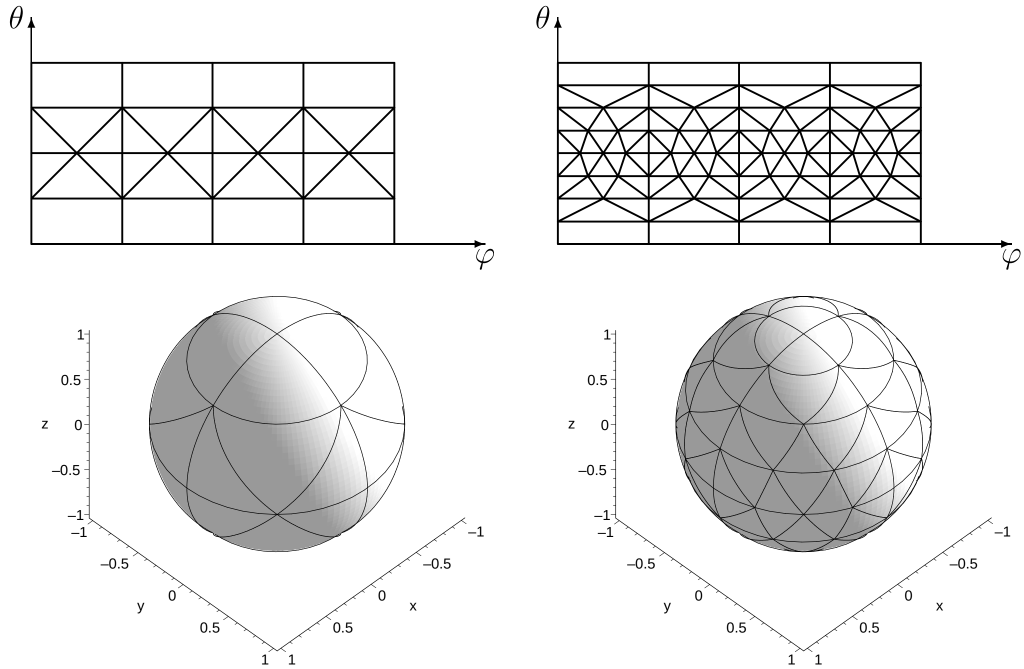


Figure 3: Isotropic triangulation of the sphere (bottom) with $n_\varphi = 4$ divisions in φ-direction and $n_\theta \in \{4, 8\}$ divisions in θ-direction and the corresponding anisotropic triangulation in the parameter domain (top)

This especially means that one edge of each element $T \in \mathcal{T}_h$ is parallel to the φ-axis in the parameter plane. Note that polar elements appear as rectangles in the parameter domain, see Remark 3.1. In this case, the element $T \subset \mathcal{S}^2$ has the nodes $(\varphi_{1,T}, \theta_{1,T})$ and $(\varphi_{2,T}, \theta_{2,T})$ (with $\theta_{1,T} = \theta_{2,T}$) and the node at the pole at $\theta = \theta_3 \in \{0, \pi\}$ and at undefined φ-coordinate. In the parameter plane, the nodes at the poles are identified, i.e. the counterpart of T in G has the nodes $(\varphi_{1,T}, \theta_{3,T})$, $(\varphi_{1,T}, \theta_{1,T})$, $(\varphi_{2,T}, \theta_{1,T})$ and $(\varphi_{2,T}, \theta_{3,T})$.

Boundary representation by edges There is a set $\mathcal{S} \subset \mathcal{E}_h$ of edges such that $\Gamma_D = \bigcup_{E \in \mathcal{S}} E$.

Isotropy Given any domain $\omega \subset G$, denote by

$$h_{\varphi,\omega} = \sup_{(\varphi,\theta)\in\omega} \varphi - \inf_{(\varphi,\theta)\in\omega} \varphi \quad \text{and} \quad h_{\theta,\omega} = \sup_{(\varphi,\theta)\in\omega} \theta - \inf_{(\varphi,\theta)\in\omega} \theta$$

its dimensions in horizontal and vertical direction, in particular $h_{\varphi,T} = \varphi_{2,T} - \varphi_{1,T}$ and $h_{\theta,T} = |\theta_{3,T} - \theta_{1,T}|$, and define

$$\vartheta_{-,\omega} := \inf_{(\varphi,\theta)\in\omega} \sin\theta, \qquad \vartheta_{+,\omega} := \sup_{(\varphi,\theta)\in\omega} \sin\theta.$$

The term $h_{\varphi,\omega}\vartheta_{+,\omega}$ corresponds to the actual horizontal extent of the domain ω on the sphere. We require

$$h_{\varphi,T}\vartheta_{+,T} \sim h_{\theta,T} \qquad \forall T \in \mathcal{T}_h, \tag{5}$$

which characterizes the isotropy of $\mathcal{T}_h$. This especially means that the φ-extent of elements at the poles is independent of h, i.e.

$$h_{\varphi,T} \sim 1 \quad \text{for } T \in \mathcal{T}_h \text{ with } \vartheta_{-,T} = 0. \tag{6}$$

Comparable size of adjacent elements We require

$$h_{\theta,T} \lesssim \vartheta_{-,T} \qquad \forall T \in \mathcal{T}_h \text{ with } \vartheta_{-,T} > 0. \tag{7}$$

This is true, for example, if adjacent elements T have approximately the same size.

Sufficient fineness The mesh generated by $\mathcal{T}_h$ is fine enough that $h_{\theta,\tilde{\omega}_T} \leq \pi/4$ at least for elements near the pole (i.e. for T with $\vartheta_{-,\tilde{\omega}_T} = 0$). Moreover, each element $T \in \mathcal{T}_h$ touches maximum one boundary corner or crack tip. For technical reasons, elements with $\vartheta_{-,\tilde{\omega}_T} = 0$ must not touch a crack tip.

Remark 3.1 *The chosen parametrization possesses a singularity at the poles of the sphere (i.e. at $\theta = 0$ and $\theta = \pi$). Resulting from this, the spherical triangles at the poles, which are produced by the previously described triangulation of Ω, correspond to rectangles in the parameter domain. "Edges" E at the poles with $\theta \equiv 0$ or $\theta \equiv \pi$ have length zero by definition,*

$$\sin\theta \equiv 0, \quad \theta = const \quad \Longrightarrow \quad \lceil E \rceil = \int_0^1 \sqrt{\dot{\varphi}^2 \sin^2\theta + \dot{\theta}^2}\,\mathrm{dt} = 0,$$

since they appear only in the parameter domain, not on the sphere. Therefore, the sets $\mathcal{E}_h$, $\mathcal{E}(T)$ etc. are always to be understood as sets of edges with $\lceil E \rceil > 0$, i.e. of spherical edges.

The set of nodes $\mathcal{N}_h$ consists of the nodes on the sphere, i.e. of all pairs of coordinates $(\varphi_{i,T}, \theta_{i,T})$ that occur for any element $T \in \mathcal{T}_h$ with $\vartheta_{-,T} > 0$ and of pole nodes, if the poles are contained in $\overline{\Omega}$. Nodes at a pole are identified in the parameter domain.

Remark 3.2 *In the following algorithm, we give an example for the generation of a mesh with all the above properties. The algorithm produces a decomposition of the parameter domain $G \subset I\!R^2$. The description is reduced to the case $\Omega = \mathcal{S}^2$. A corresponding triangulation of proper subdomains of the unit sphere can be found by choosing the parameters properly and adapting the mesh to the corresponding parameter domain.*

Algorithm Suppose $G = [0, 2\pi) \times [0, \pi]$.

- The parameter domain G is divided n_φ times in φ-direction and n_θ times in θ-direction, where n_φ is about 4 or 6 and where n_θ is even.
- The generated rectangles at $\theta = 0$ and $\theta = \pi$ remain unchanged. They correspond to the triangles at the poles.

  Let $h_\theta = \pi/n_\theta$. All edges with $\theta = k \cdot h_\theta \leq \pi/2$, $k = 1, \ldots, n_\theta/2$, are divided equidistantly into $k-1$ parts.
- The generated nodes are connected properly with each other and the corresponding end points of the edges. The generated mesh is mirrored on the equator ($\theta \equiv \pi/2$), see Figure 3.

Lemma 3.3 *The relation*

$$\vartheta_{+,T} \sim \vartheta_{-,T} \tag{8}$$

holds true for all elements $T \in \mathcal{T}_h$ which are not placed at a pole, i.e. for $\vartheta_{-,T} > 0$.

Proof Relation (8) was proven already in [3, Subsection 4.3]. For completeness, the proof is repeated. It is clear that $\vartheta_{+,T} \geq \vartheta_{-,T}$. It remains to show $\vartheta_{+,T} \lesssim \vartheta_{-,T}$.

By definition, there are two angles θ_- and θ_+ with $\vartheta_{-,T} = \sin\theta_-$ and $\vartheta_{+,T} = \sin\theta_+$. Obviously, we have $|\theta_+ - \theta_-| \leq h_{\theta,T}$. Because of assumption (7) and $\sin x \leq |x|\ \forall x \in I\!R$ it follows that

$$\vartheta_{-,T}^{-1} \lesssim h_{\theta,T}^{-1}, \qquad \sin(\theta_+ - \theta_-) \leq |\theta_+ - \theta_-| \leq h_{\theta,T},$$

and therefore

$$\begin{aligned} \vartheta_{+,T} &= \sin\theta_+ = \sin\theta_- \cos(\theta_+ - \theta_-) + \cos\theta_- \sin(\theta_+ - \theta_-) \\ &= \vartheta_{-,T}\Big(\cos(\theta_+ - \theta_-) + \vartheta_{-,T}^{-1} \cos\theta_- \sin(\theta_+ - \theta_-)\Big) \\ &\lesssim \vartheta_{-,T}. \end{aligned}$$

■

Corollary 3.4 *Let $T \in \mathcal{T}_h$ with $\vartheta_{-,T} > 0$. Relation (5) and the previous lemma imply*

$$\sin\theta \sim \frac{h_{\theta,T}}{h_{\varphi,T}} \quad \forall (\varphi, \theta) \in T. \tag{9}$$

Remark 3.5 *The Properties (5), (6) and (9) yield*

$$\lceil T \rceil \sim h_{\theta,T}^2 \quad \forall T \in \mathcal{T}_h \quad \textit{and} \quad \lceil E \rceil \sim h_{\theta,T} \quad \forall E \in \mathcal{E}_h,\ \forall T \subset \omega_E. \tag{10}$$

This especially gives

$$h_{\theta,T_1} \sim h_{\theta,T_2} \quad \textit{for any two adjacent elements } T_1, T_2 \in \mathcal{T}_h. \tag{11}$$

Corresponding to the triangulation $\mathcal{T}_h$ of Ω, let X_h be the space of all continuous, elementwise linear finite element functions that vanish on Γ_D. Here, *linear* means *linear with respect to φ and θ*, i.e. linear over the parameter domain G. Since pole triangles appear as rectangles in the parameter domain, the finite element functions of X_h must be bilinear over these elements. In particular, if $v \in X_h$, then let $v|_T \in \operatorname{span}\{1, \varphi, \theta\}$, if $\vartheta_{-,T} > 0$, and let $v|_T \in \operatorname{span}\{1, \theta, \varphi\theta\}$, if $\vartheta_{-,T} = 0$, see also Remark 5.1 for formulae of the basis functions.

The finite element discretization of Problem (3) is given by: Find $u_h \in X_h$ such that

$$\int\limits_G \nabla_{\mathcal{S}} u_h \cdot \nabla_{\mathcal{S}} v_h \, \mathrm{d}\omega = \int\limits_G f v_h \, \mathrm{d}\omega + \int\limits_{\Gamma_N} g v_h \, \mathrm{d}\sigma \quad \forall v_h \in X_h, \tag{12}$$

where $\mathrm{d}\omega = \sin\theta \, \mathrm{d}\varphi \, \mathrm{d}\theta$ and $\mathrm{d}\sigma = \sqrt{\dot{\varphi}^2 \sin^2\theta(t) + \dot{\theta}^2}\, \mathrm{dt}$ are the spherical surface and line elements, see Subsection 2.2.

4 Basic estimates for spherical domains

In this section, basic estimates are summarized which are necessary for the verification of forthcoming error estimates. Their proofs are based on transformations to reference domains, conversion of norms and application of corresponding estimates for plane domains.

Lemma 4.1 (Trace theorem) *Let $T \in \mathcal{T}_h$ and $E \in \mathcal{E}(T)$. Then all $v \in \mathcal{H}^1(T)$ satisfy*

$$\lVert v \rVert_{0,E}^2 \lesssim \frac{\lceil E \rceil}{\lceil T \rceil} \Big(\lVert v \rVert_{0,T}^2 + h_{\theta,T}^2 \lceil v \rceil_{1,T}^2 \Big).$$

The proof of Lemma 4.1 is given in Appendix A.

Lemma 4.2 (Bramble-Hilbert) *Let $T \in \mathcal{T}_h$, $E \in \mathcal{E}(T)$, $x \in \mathcal{N}(T)$ and $\omega = T$, $\omega = \omega_x$, $\omega = \tilde{\omega}_T$ or $\omega = \tilde{\omega}_E$. For all functions $v \in \mathcal{H}^1(\omega)$ there is a constant function $\tau = \tau(v) \in \mathcal{P}_{0|\omega}$, such that*

$$\lVert v - \tau \rVert_{0,\omega} \lesssim h_{\theta,T} \lceil v \rceil_{1,\omega}.$$

The proof of Lemma 4.2 is given in Appendix B. It refers to a similar estimate stated by Dupont, Scott [8], see also Brenner, Scott [4], where $\tau(v)$ is chosen using averaged Taylor polynomials.

Let $\omega \subset \Omega$ and let $\mathcal{P}_0$ be the space of all constant functions over Ω. We denote by $\pi_{0,\omega} : \mathcal{H}^0(\omega_x) \to \mathcal{P}_0$ the *weighted* L^2-projection of a function $v \in \mathcal{H}^0(\omega)$ onto $\mathcal{P}_0$, i.e. let

$$\pi_{0,\omega} v = \frac{1}{\lceil \omega \rceil} \int\limits_\omega v(\varphi, \theta) \sin\theta \, \mathrm{d}\varphi \, \mathrm{d}\theta = \lVert 1 \rVert_{0,\omega}^{-2} \int\limits_\omega v \, \mathrm{d}\omega.$$

Corollary 4.3 (Error estimate for the weighted L^2-projection) *Let $T \in \mathcal{T}_h$, $E \in \mathcal{E}(T)$, $x \in \mathcal{N}(T)$ and let ω be any of the domains considered in Lemma 4.2. Denote by $\pi_{0,\omega}$ the* weighted *L^2-projection of a function v onto $\mathcal{P}_0$. Then the estimate*

$$\lceil\!| v - \pi_{0,\omega} v |\!\rceil_{0,\omega} \lesssim h_{\theta,T} \lceil v \rceil_{1,\omega}. \tag{13}$$

holds for all $v \in \mathcal{H}^1(\omega)$.

Corollary 4.3 complies with [4, Lemma 4.3.14] where the result is called Friedrichs' inequality. For completeness the proof is given below.

Proof According to Lemma 4.2 there is a constant function $\tau(v)$ depending on v such that

$$\lceil\!| v - \tau(v) |\!\rceil_{0,\omega} \lesssim h_{\theta,T} \lceil v \rceil_{1,\omega}.$$

Moreover the relation $\pi_{0,\omega}\tau(v) = \tau(v)$ holds true, since $\tau(v)$ is constant over ω. From the definition of $\pi_{0,\omega}$, one obtains for any function $\psi \in H^0(\omega)$

$$\begin{aligned} |\pi_{0,\omega}\psi| &= \lceil\!| 1 |\!\rceil_{0,\omega}^{-2} \int_\omega \psi \, \mathrm{d}\omega \;\leq\; \lceil\!| 1 |\!\rceil_{0,\omega}^{-2} \lceil\!| \psi |\!\rceil_{0,\omega} \lceil\!| 1 |\!\rceil_{0,\omega} \;=\; \lceil\!| 1 |\!\rceil_{0,\omega}^{-1} \lceil\!| \psi |\!\rceil_{0,\omega}, \\ \lceil\!| \pi_{0,\omega}\psi |\!\rceil_{0,\omega} &= |\pi_{0,\omega}\psi| \lceil\!| 1 |\!\rceil_{0,T} \;\leq\; \lceil\!| 1 |\!\rceil_{0,\omega}^{-1} \lceil\!| \psi |\!\rceil_{0,\omega} \lceil\!| 1 |\!\rceil_{0,\omega} \;\leq\; \lceil\!| \psi |\!\rceil_{0,\omega}. \end{aligned} \tag{14}$$

Choosing $\psi = v - \tau(v)$ and applying Lemma 4.2, we get

$$\begin{aligned} \lceil\!| v - \pi_{0,\omega} v |\!\rceil_{0,\omega} &= \lceil\!| (v - \tau(v)) - \pi_{0,\omega}(v - \tau(v)) |\!\rceil_{0,\omega} \\ &\leq \lceil\!| v - \tau(v) |\!\rceil_{0,\omega} + \lceil\!| \pi_{0,\omega}(v - \tau(v)) |\!\rceil_{0,\omega} \\ &\leq 2\lceil\!| v - \tau(v) |\!\rceil_{0,\omega} \;\lesssim\; h_{\theta,T} \lceil v \rceil_{1,\omega}. \end{aligned}$$

■

5 Interpolation in spherical domains

We define the (weighted) Clément-type interpolation operator $I_h : \mathcal{H}^0(\Omega) \mapsto X_h$ by:

$$I_h v(\varphi, \theta) = \sum_{x \in \mathcal{N}_h \setminus \mathcal{N}_{h,D}} (\pi_{0,\omega_x} v) \; \phi_x(\varphi, \theta),$$

where the nodal basis functions ϕ_x are piecewise affine linear (or bilinear) with respect to φ and θ, i.e. piecewise polynomials of first degree over the parameter domain, see Remark 5.1.

Remark 5.1 *The nodal basis functions ϕ_x have the value 1 in the node $x \in \mathcal{N}_h$ and the value 0 in all other nodes. They are defined over the patch ω_x and extended by zero on $\Omega \setminus \omega_x$. For elements $T \in \mathcal{T}_h$ with $\vartheta_{-,T} > 0$, we can choose, for example,*

$$\begin{aligned} \phi_{x_{1,T}} &= 1 - \wp^{-1}[(\varphi - \varphi_{1,T})(\theta_{3,T} - \theta_{2,T}) - (\varphi_{3,T} - \varphi_{2,T})(\theta - \theta_{1,T})], \\ \phi_{x_{2,T}} &= \wp^{-1}[(\varphi - \varphi_{1,T})(\theta_{3,T} - \theta_{1,T}) - (\varphi_{3,T} - \varphi_{1,T})(\theta - \theta_{1,T})], \\ \phi_{x_{3,T}} &= \wp^{-1}[(\varphi_{2,T} - \varphi_{1,T})(\theta - \theta_{1,T}) - (\varphi - \varphi_{1,T})(\theta_{2,T} - \theta_{1,T})], \end{aligned}$$

where $\wp = (\varphi_{2,T} - \varphi_{1,T})(\theta_{3,T} - \theta_{1,T}) - (\varphi_{3,T} - \varphi_{1,T})(\theta_{2,T} - \theta_{1,T})$. Exploiting Relation (4), these terms can be simplified. As already mentioned in Section 3, polar elements have to be considered separately. By assumption, see Section 3 and Remark 3.1, elements $T \in \mathcal{T}_h$ with $\vartheta_{-,T} = 0$, have the nodes $x_{1,T} = (\varphi_{1,T}, \theta_{1,T})$, $x_{2,T} = (\varphi_{2,T}, \theta_{1,T})$ and $x_{3,T}$ at $\theta = \theta_3 \in \{0, \pi\}$, i.e. $\mathcal{N}(T) = \{x_{1,T}, x_{2,T}, x_{3,T}\}$. For these elements, the nodal basis functions are allowed to be bilinear. They can be chosen, for example, as follows:

$$\begin{aligned}
\phi_{x_{1,T}} &= \wp^{-1}(\varphi_{2,T} - \varphi)(\theta_{3,T} - \theta), \\
\phi_{x_{2,T}} &= \wp^{-1}(\varphi - \varphi_{1,T})(\theta_{3,T} - \theta), \\
\phi_{x_{3,T}} &= \wp^{-1}(\varphi_{2,T} - \varphi_{1,T})(\theta - \theta_{1,T})
\end{aligned}$$

with the scaling factor $\wp = (\varphi_{2,T} - \varphi_{1,T})(\theta_{3,T} - \theta_{1,T})$.

One will check that all given basis functions have positive values over the corresponding element T with maximum value 1 and that

$$\sum_{x \in \mathcal{N}(T)} \phi_x = 1 \quad \textit{for all} \quad T \in \mathcal{T}_h.$$

Moreover, constant functions are interpolated exactly over elements $T \in \mathcal{T}_h$ which have no nodes on the Dirichlet boundary, i.e. $I_h v = v$ for $v \in \mathcal{P}_{0|T}$, $T \in \mathcal{T}_h$, $\partial T \cap \Gamma_D = 0$.

The definition of the interpolation operator implies that $I_h v = 0$ on Γ_D.

Remark 5.2 *The nodal basis functions for elements $T \in \mathcal{T}_h$ with $\vartheta_{-,T} > 0$ equal the barycentric coordinates corresponding to T (considered as triangle in the parameter plane). One will check that the nodal basis functions for elements $T \in \mathcal{T}_h$ with $\vartheta_{-,T} = 0$ (see Remark 5.1) have the same properties as barycentric coordinates, especially $\phi_{x_{1,T}}\phi_{x_{2,T}}\phi_{x_{3,T}} \leq 1/27$, where the maximum value $1/27$ is attained at $\varphi = 1/2(\varphi_{1,T} + \varphi_{2,T})$, $\theta = 1/3(2\theta_{1,T} + \theta_{3,T})$, and $\phi_{x_{i,T}}\phi_{x_{j,T}} \leq 1/4$ for $i \neq j$ with maximum value $1/4$.*

Lemma 5.3 *The interpolation operator I_h is bounded, i.e.*

$$\begin{aligned}
\lceil\!\lceil I_h v \rceil\!\rceil_{0,T} &\leq \lceil\!\lceil v \rceil\!\rceil_{0,\tilde{\omega}_T} && \forall T \in \mathcal{T}_h, && (15) \\
\lceil\!\lceil I_h v \rceil\!\rceil_{0,E} &\lesssim h_{\theta,T}^{-1/2} \lceil\!\lceil v \rceil\!\rceil_{0,\tilde{\omega}_E} && \forall E \in \mathcal{E}(T),\ T \in \mathcal{T}_h. && (16)
\end{aligned}$$

Proof From (14) we know that for any $T \in \mathcal{T}_h$ and any $x \in \mathcal{N}(T)$

$$|\pi_{0,\omega_x} v| \ \leq\ \lceil\!\lceil 1 \rceil\!\rceil_{0,\omega_x}^{-1} \lceil\!\lceil v \rceil\!\rceil_{0,\omega_x} \ \leq\ \lceil\!\lceil 1 \rceil\!\rceil_{0,T}^{-1} \lceil\!\lceil v \rceil\!\rceil_{0,\tilde{\omega}_T} \ =\ \lceil T \rceil^{-1/2} \lceil\!\lceil v \rceil\!\rceil_{0,\tilde{\omega}_T}$$

and from the properties of the basis functions ϕ_x that

$$\sum_{x \in \mathcal{N}(T) \setminus \mathcal{N}_{h,D}} \phi_x(\varphi, \theta) \leq 1 \qquad \forall (\varphi, \theta) \in T.$$

This gives

$$\begin{aligned}
\lceil\!\lceil I_h v \rceil\!\rceil_{0,T}^2 &= \int_T |I_h v|^2 \,\mathrm{d}\omega \ \leq\ \int_T \Big\{ \sum_{x \in \mathcal{N}(T) \setminus \mathcal{N}_{h,D}} |\pi_{0,\omega_x} v|\ \phi_x \Big\}^2 \mathrm{d}\omega \\
&\leq \lceil T \rceil^{-1} \lceil\!\lceil v \rceil\!\rceil_{0,\tilde{\omega}_T}^2 \int_T \Big\{ \sum_{x \in \mathcal{N}(T) \setminus \mathcal{N}_{h,D}} \phi_x \Big\}^2 \mathrm{d}\omega \\
&\leq \lceil T \rceil^{-1} \lceil\!\lceil v \rceil\!\rceil_{0,\tilde{\omega}_T}^2 \lceil T \rceil \ =\ \lceil\!\lceil v \rceil\!\rceil_{0,\tilde{\omega}_T}^2,
\end{aligned}$$

which proves relation (15). Now let $x_{0,E} = (\varphi(0), \theta(0))$ and $x_{1,E} = (\varphi(1), \theta(1))$ denote the vertices of $E \in \mathcal{E}(T)$. Using (14), we have for $\alpha \in \{0, 1\}$

$$\Big(I_h v(\alpha)\Big)^2 = \Big(I_h v(x_{\alpha,E})\Big)^2 = |\pi_{0,\omega_{x_{\alpha,E}}} v|^2 \leq \lceil T\rceil^{-1/2} \lVert\lceil v\rceil\rVert_{0,\tilde{\omega}_E}.$$

This yields for $t \in [0, 1]$

$$\begin{aligned}
\Big(I_h v(t)\Big)^2 &\leq \Big(\max_{t\in[0,1]} |I_h v(t)|\Big)^2 \leq \Big(\max\{|I_h v(0)|, |I_h v(1)|\}\Big)^2 \leq \Big(I_h v(0)\Big)^2 + \Big(I_h v(1)\Big)^2 \\
&\leq 2\lceil T\rceil^{-1/2} \lVert\lceil v\rceil\rVert_{0,\tilde{\omega}_E}.
\end{aligned}$$

The last term is independent of t. Hence, we use (10) and continue

$$\lVert\lceil I_h v\rceil\rVert_{0,E}^2 = \int_E \Big(I_h v(t)\Big)^2 \,\mathrm{d}\sigma \leq 2\lceil E\rceil \lceil T\rceil^{-1} \lVert\lceil v\rceil\rVert_{0,\tilde{\omega}_T} \sim h_{\theta,T}^{-1} \lVert\lceil v\rceil\rVert_{0,\tilde{\omega}_T},$$

which proves the second inequality. ■

Theorem 5.4 (Interpolation Error Estimates) *Let $\mathcal{T}_h$ be an isotropic triangulation of a regular spherical domain. Then the following interpolation error estimates hold true:*

$$\begin{aligned}
\lVert\lceil v - I_h v\rceil\rVert_{0,T} &\lesssim h_{\theta,T} \lceil v\rceil_{1,\tilde{\omega}_T} \quad \forall v \in \mathcal{H}^1(\tilde{\omega}_T), \\
\lVert\lceil v - I_h v\rceil\rVert_{0,E} &\lesssim h_{\theta,T}^{1/2} \lceil v\rceil_{1,\tilde{\omega}_E} \quad \forall v \in \mathcal{H}^1(\tilde{\omega}_E).
\end{aligned}$$

Proof By definition, constant functions $w \in \mathcal{P}_{0|\tilde{\omega}_T}$ are interpolated exactly over elements $T \in \mathcal{T}_h$ which do not collide with the Dirichlet boundary, i.e. if $\mathcal{N}(T) \cap \mathcal{N}_{h,D} = \emptyset$. In this case, we get from the triangle inequality

$$\lVert\lceil v - I_h v\rceil\rVert_{0,T} \leq \lVert\lceil v - w\rceil\rVert_{0,T} + \lVert\lceil I_h(v - w)\rceil\rVert_{0,T} \qquad \forall w \in \mathcal{P}_{0|\tilde{\omega}_T}.$$

Lemma 5.3 and Lemma 4.2 imply the existence of a function $w \in \mathcal{P}_{0|\tilde{\omega}_T}$ with $\lVert\lceil v - I_h v\rceil\rVert_{0,T} \lesssim \lVert\lceil v - w\rceil\rVert_{0,\tilde{\omega}_T} \lesssim h_{\theta,T} \lceil v\rceil_{1,\tilde{\omega}_T}$.

If $\mathcal{N}(T) \cap \mathcal{N}_{h,D} \neq \emptyset$, we have

$$\begin{aligned}
\lVert\lceil v - I_h v\rceil\rVert_{0,T}^2 &= \int_T \Big\{v - \sum_{x\in\mathcal{N}(T)\setminus\mathcal{N}_{h,D}} \pi_{0,\omega_x} v\ \phi_x\Big\}^2 \mathrm{d}\omega \\
&= \int_T \Big\{v - \sum_{x\in\mathcal{N}(T)} \pi_{0,\omega_x} v\ \phi_x + \sum_{x\in\mathcal{N}(T)\cap\mathcal{N}_{h,D}} \pi_{0,\omega_x} v\ \phi_x\Big\}^2 \mathrm{d}\omega \\
&\leq 4 \int_T \Big\{v - \sum_{x\in\mathcal{N}(T)} \pi_{0,\omega_x} v\ \phi_x\Big\}^2 + \sum_{x\in\mathcal{N}(T)\cap\mathcal{N}_{h,D}} \Big\{\pi_{0,\omega_x} v\ \phi_x\Big\}^2 \mathrm{d}\omega \\
&\leq 4\Bigg(\lVert\lceil v - \sum_{x\in\mathcal{N}(T)} \pi_{0,\omega_x} v\ \phi_x\rceil\rVert_{0,T}^2 + \sum_{x\in\mathcal{N}(T)\cap\mathcal{N}_{h,D}} \lVert\lceil \pi_{0,\omega_x} v\ \phi_x\rceil\rVert_{0,T}^2\Bigg).
\end{aligned}$$

In the first term, we insert a constant function w and proceed as in the first part of this proof by applying Lemma 4.2. Using $\phi_x(\varphi, \theta) \leq 1$ for all $x \in \mathcal{N}(T)$, $(\varphi, \theta) \in \omega_x$, and consequently $\lVert\lceil \pi_{0,\omega_x} v \phi_x\rceil\rVert_{0,T}^2 \leq \lVert\lceil \pi_{0,\omega_x} v\rceil\rVert_{0,T}^2 = \lceil T\rceil |\pi_{0,\omega_x} v|^2$, one obtains

$$\lVert\lceil v - I_h v\rceil\rVert_{0,T}^2 \lesssim h_{\theta,T}^2 \lceil v\rceil_{1,\tilde{\omega}_T}^2 + \lceil T\rceil \sum_{x\in\mathcal{N}(T)\cap\mathcal{N}_{h,D}} |\pi_{0,\omega_x} v|^2. \tag{17}$$

For the estimation of the second term, we exploit that for each $x \in \mathcal{N}_{h,D}$, there is an edge $E \subset \Gamma_D$ and an element $T' \subset \omega_x$ with $E \in \mathcal{E}(T')$ and $v \equiv 0$ on E. Because $\pi_{0,\omega_x} v$ is constant over ω_x and v vanishes on E, we have by using the trace theorem (Lemma 4.1)

$$\begin{aligned} |\pi_{0,\omega_x} v|^2 &= \lceil E \rceil^{-1} \| \pi_{0,\omega_x} v \|_{0,E}^2 = \lceil E \rceil^{-1} \| v - \pi_{0,\omega_x} v \|_{0,E}^2 \\ &\lesssim \lceil T' \rceil^{-1} \Big(\| v - \pi_{0,\omega_x} v \|_{0,T'}^2 + h_{\theta,T'}^2 \lceil v - \pi_{0,\omega_x} v \rceil_{1,T'}^2 \Big). \end{aligned}$$

The size of adjacent elements does not change rapidly, i.e. $h_{\theta,T'} \sim h_{\theta,T}$ and $\lceil T' \rceil \sim \lceil T \rceil$ for all $T' \subset \tilde{\omega}_T$. Moreover, $\lceil v - \pi_{0,\omega_x} v \rceil_{1,T'} = \lceil v \rceil_{1,T'}$ holds, since $\pi_{0,\omega_x} v \in \mathcal{P}_0$. Combining all these facts and Estimate (13), we derive

$$|\pi_{0,\omega_x} v|^2 \lesssim \lceil T \rceil^{-1} (h_{\theta,T}^2 \lceil v \rceil_{1,\omega_x}^2 + h_{\theta,T}^2 \lceil v \rceil_{1,T'}^2) \sim \lceil T \rceil^{-1} h_{\theta,T}^2 \lceil v \rceil_{1,\omega_x}^2 .$$

Inserting this into (17) gives

$$\| v - I_h v \|_{0,T}^2 \lesssim h_{\theta,T}^2 \lceil v \rceil_{1,\tilde{\omega}_T}^2 + \sum_{x \in \mathcal{N}(T) \cap \mathcal{N}_{h,D}} h_{\theta,T}^2 \lceil v \rceil_{1,\omega_x}^2 \lesssim h_{\theta,T}^2 \lceil v \rceil_{1,\tilde{\omega}_T}^2 .$$

Now let $T \in \mathcal{T}_h$ and $E \in \mathcal{E}(T)$ with $\mathcal{N}(E) \cap \mathcal{N}_{h,D} = \emptyset$. Each $w \in \mathcal{P}_0$ satisfies $\partial w / \partial \varphi = 0$ and $\partial w / \partial \theta = 0$. The spherical trace theorem (Lemma 4.1) implies for all $w \in \mathcal{P}_0$

$$\| v - w \|_{0,E} \lesssim \frac{\lceil E \rceil}{\lceil T \rceil} \Big(\| v - w \|_{0,\tilde{\omega}_E}^2 + h_{\theta,T}^2 \lceil v \rceil_{1,\tilde{\omega}_E}^2 \Big)^{1/2},$$

and thus Relations (16) and (10) provide

$$\begin{aligned} \| v - I_h v \|_{0,E} &\leq \| v - w \|_{0,E} + \| I_h (v - w) \|_{0,E} \\ &\lesssim h_{\theta,T}^{-1/2} \Big(\| v - w \|_{0,\tilde{\omega}_E}^2 + h_{\theta,T}^2 \lceil v \rceil_{1,\tilde{\omega}_E}^2 \Big)^{1/2} + h_{\theta,T}^{-1/2} \| v - w \|_{0,\tilde{\omega}_E}. \end{aligned}$$

The assertion then follows from Lemma 4.2.

In the case $\mathcal{N}(E) \cap \mathcal{N}_{h,D} \neq \emptyset$, we proceed as for triangles. With similar arguments as above we can estimate

$$\begin{aligned} \| v - I_h v \|_{0,E}^2 &\lesssim \| v - \sum_{x \in \mathcal{N}(E)} \pi_{0,\omega_x} v \phi_x \|_{0,E}^2 + \sum_{x \in \mathcal{N}(E) \cap \mathcal{N}_{h,D}} \| \pi_{0,\omega_x} v \|_{0,E}^2 \\ &\lesssim h_{\theta,T} \lceil v \rceil_{1,\tilde{\omega}_E}^2 + \lceil E \rceil \sum_{x \in \mathcal{N}(E) \cap \mathcal{N}_{h,D}} |\pi_{0,\omega_x} v|^2 \\ &\lesssim h_{\theta,T} \lceil v \rceil_{1,\tilde{\omega}_E}^2 + \lceil E \rceil \lceil T \rceil^{-1} \sum_{x \in \mathcal{N}(E) \cap \mathcal{N}_{h,D}} h_{\theta,T}^2 \lceil v \rceil_{1,\omega_x}^2 \\ &\lesssim h_{\theta,T} \lceil v \rceil_{1,\tilde{\omega}_E}^2 . \end{aligned}$$

■

6 A residual error estimator

The aim of this section is to prove a reliable and efficient error estimator for the mixed boundary value problem for the Laplace-Beltrami operator described in (1). The weak formulation and finite element discretization are given by Formulae (3) and (12). For simplification, we assume in the following that we can evaluate integrals exactly.

With every edge $E \in \mathcal{E}_h$ and each $\mathbf{x} \in E$, a unit vector $\mathbf{n}_E(\mathbf{x})$ is associated such that $\mathbf{n}_E(\mathbf{x})$ is orthogonal to the tangential vector on the curve $E \subset \mathcal{S}^2$ at $\mathbf{x}$ and such that $\mathbf{n}_E(\mathbf{x})$ lies in the tangential plane at $\mathbf{x}$. For boundary edges $E \subset \partial\Omega$, this vector shall equal the exterior normal vector to $\partial\Omega$. For any interior edge $E \in \mathcal{E}_{h,\Omega}$, the jump of a function ψ with $\psi\mid_{T'} \in C(T')$ for all $T' \subset \omega_E$ across E in direction $\mathbf{n}_E$ is defined by

$$[\psi]_E(\mathbf{x}) := \lim_{t\to 0+} \psi(\mathbf{x} + t\mathbf{n}_E(\mathbf{x})) - \lim_{t\to 0+} \psi(\mathbf{x} - t\mathbf{n}_E(\mathbf{x})).$$

Denote by $\mathbf{n}_{E,T}$ the exterior normal to ∂T on E. One checks that for vector functions $\mathcal{U}$ and scalar functions v with $v \equiv 0$ on Γ_D the relation

$$\sum_{T\in\mathcal{T}_h} \sum_{E\in\mathcal{E}(T)} \int_E \{\mathbf{n}_{E,T} \cdot \mathcal{U}\} v \,\mathrm{d}\sigma = \sum_{T\in\mathcal{T}_h} \sum_{E\in\mathcal{E}_{h,N}} \int_E \{\mathbf{n}_E \cdot \mathcal{U}\} v \,\mathrm{d}\sigma - \sum_{E\in\mathcal{E}_{h,\Omega}} \int_E [\mathbf{n}_E \cdot \mathcal{U}]_E v \,\mathrm{d}\sigma \quad (18)$$

holds, where we assume that all integrals exist.

6.1 Reliability

The space X_h, defined in Section 3, is a subset of $X = \{v \in \mathcal{H}^1(\Omega) \mid v = 0 \text{ on } \Gamma_D\}$. Hence, the Galerkin orthogonality of $u - u_h$ follows from (3) and (12), i.e. the error is orthogonal to X_h:

$$\int_G \nabla_{\mathcal{S}}(u - u_h) \cdot \nabla_{\mathcal{S}} v_h \,\mathrm{d}\omega = 0 \quad \forall v_h \in X_h. \quad (19)$$

Assume that $u_h \in X_h$ is the exact solution of Problem (12). Elementwise integration by parts and Relation (18) provide

$$\begin{aligned}
\int_G \nabla_{\mathcal{S}} u_h \cdot \nabla_{\mathcal{S}} v \,\mathrm{d}\omega &= \sum_{T\in\mathcal{T}_h} \int_T \nabla_{\mathcal{S}} u_h \cdot \nabla_{\mathcal{S}} v \,\mathrm{d}\omega \\
&= \sum_{T\in\mathcal{T}_h} \Big\{ \int_T -\Delta_{\mathcal{S}} u_h v \,\mathrm{d}\omega + \sum_{E\in\mathcal{E}(T)} \int_E \mathbf{n}_{E,T} \cdot \nabla_{\mathcal{S}} u_h v \,\mathrm{d}\sigma \Big\} \\
&= -\sum_{T\in\mathcal{T}_h} \int_T \Delta_{\mathcal{S}} u_h v \,\mathrm{d}\omega + \sum_{E\in\mathcal{E}_{h,N}} \int_E \mathbf{n}_E \cdot \nabla_{\mathcal{S}} u_h v \,\mathrm{d}\sigma - \sum_{E\in\mathcal{E}_{h,\Omega}} \int_E [\mathbf{n}_E \cdot \nabla_{\mathcal{S}} u_h]_E v \,\mathrm{d}\sigma.
\end{aligned}$$

The combination with (3) gives

$$\begin{aligned}
\int_G \nabla_{\mathcal{S}}(u - u_h) \cdot \nabla_{\mathcal{S}} v \,\mathrm{d}\omega &= \int_G f v \,\mathrm{d}\omega + \int_{\Gamma_N} g v \,\mathrm{d}\sigma - \int_G \nabla_{\mathcal{S}} u_h \cdot \nabla_{\mathcal{S}} v \,\mathrm{d}\omega \\
&= \sum_{T\in\mathcal{T}_h} \int_T \{f + \Delta_{\mathcal{S}} u_h\} v \,\mathrm{d}\omega + \sum_{E\in\mathcal{E}_{h,N}} \int_E \{g - \mathbf{n}_E \cdot \nabla_{\mathcal{S}} u_h\} v \,\mathrm{d}\sigma \\
&\quad + \sum_{E\in\mathcal{E}_{h,\Omega}} \int_E [\mathbf{n}_E \cdot \nabla_{\mathcal{S}} u_h]_E v \,\mathrm{d}\sigma. \quad (20)
\end{aligned}$$

For abbreviation we introduce now

$$R_T := \{f + \Delta_{\mathcal{S}} u_h\}|_T, \quad R_E := [\mathbf{n}_E \cdot \nabla_{\mathcal{S}} u_h]_E \quad \text{and} \quad R_N := g_E - \mathbf{n}_E \cdot \nabla_{\mathcal{S}} u_h. \tag{21}$$

By definition, we have $I_h v \in X_h$ for all $v \in X$ and can apply the Galerkin orthogonality (19), i.e. we can replace v by $v - I_h v$ in (20). From the interpolation error estimates of Theorem 5.4 and the Cauchy-Schwarz inequality, one obtains

$$\begin{aligned}
\int_G \nabla_{\mathcal{S}}(u-u_h)\cdot\nabla_{\mathcal{S}} v \,\mathrm{d}\omega &= \int_G \nabla_{\mathcal{S}}(u-u_h)\cdot\nabla_{\mathcal{S}}(v - I_h v)\,\mathrm{d}\omega \\
&= \sum_{T\in\mathcal{T}_h}\int_T R_T(v-I_hv)\,\mathrm{d}\omega + \sum_{E\in\mathcal{E}_{h,N}}\int_E R_N(v-I_hv)\,\mathrm{d}\sigma + \sum_{E\in\mathcal{E}_{h,\Omega}}\int_E R_E(v-I_hv)\,\mathrm{d}\sigma \\
&\lesssim \sum_{T\in\mathcal{T}_h} h_{\theta,T}\lceil\!\lceil R_T\rceil\!\rceil_{0,T}\lceil v\rceil_{1,\tilde\omega_T} + \sum_{E\in\mathcal{E}_{h,N}} h_{\theta,T}^{1/2}\lceil\!\lceil R_N\rceil\!\rceil_{0,E}\lceil v\rceil_{1,\tilde\omega_E} + \sum_{E\in\mathcal{E}_{h,\Omega}} h_{\theta,T}^{1/2}\lceil\!\lceil R_E\rceil\!\rceil_{0,E}\lceil v\rceil_{1,\tilde\omega_E} \\
&\le \Big\{\sum_{T\in\mathcal{T}_h} h_{\theta,T}^2\lceil\!\lceil R_T\rceil\!\rceil_{0,T}^2 + \sum_{E\in\mathcal{E}_{h,N}} h_{\theta,T}\lceil\!\lceil R_N\rceil\!\rceil_{0,E}^2 + \sum_{E\in\mathcal{E}_{h,\Omega}} h_{\theta,T}\lceil\!\lceil R_E\rceil\!\rceil_{0,E}^2\Big\}^{1/2}\cdot \\
&\quad \cdot\Big\{\sum_{T\in\mathcal{T}_h}\lceil v\rceil_{1,\tilde\omega_T}^2 + \sum_{E\in\mathcal{E}_{h,N}\cup\mathcal{E}_{h,\Omega}}\lceil v\rceil_{1,\tilde\omega_E}^2\Big\}^{1/2} \\
&\lesssim \lceil v\rceil_{1,G}\Big\{\sum_{T\in\mathcal{T}_h} h_{\theta,T}^2\lceil\!\lceil R_T\rceil\!\rceil_{0,T}^2 + \sum_{E\in\mathcal{E}_{h,N}} h_{\theta,T}\lceil\!\lceil R_N\rceil\!\rceil_{0,E}^2 + \sum_{E\in\mathcal{E}_{h,\Omega}} h_{\theta,T}\lceil\!\lceil R_E\rceil\!\rceil_{0,E}^2\Big\}^{1/2},
\end{aligned}$$

where the last inequality follows from Condition (iii) on the triangulation; that is, there is only a limited number of patches ω_T or ω_E containing one and the same element T'. This estimate holds for all $v \in X$, especially for $v = u - u_h$. Hence, division by $\lceil u - u_h\rceil_{1,G}$ yields

$$\lceil u-u_h\rceil_{1,G} \lesssim \Big\{\sum_{T\in\mathcal{T}_h} h_{\theta,T}^2\lceil\!\lceil R_T\rceil\!\rceil_{0,T}^2 + \sum_{E\in\mathcal{E}_{h,N}} h_{\theta,T}\lceil\!\lceil R_N\rceil\!\rceil_{0,E}^2 + \sum_{E\in\mathcal{E}_{h,\Omega}} h_{\theta,T}\lceil\!\lceil R_E\rceil\!\rceil_{0,E}^2\Big\}^{1/2}. \tag{22}$$

The right hand side can be used as an error estimator. In order to obtain a lower bound for the error, see Subsection 6.2, residuals from a finite dimensional space are necessary (cf. standard textbooks, e.g. Verfürth [17]). To this end, we define the $\mathcal{L}^2$-projections

$$\begin{aligned}
\pi_{0,T}R_T &:= \frac{1}{\lceil T\rceil}\int_T R_T\,\mathrm{d}\omega \quad \text{for } T\in\mathcal{T}_h, \\
\pi_{0,E}R_E &:= \frac{1}{\lceil E\rceil}\int_E R_E\,\mathrm{d}\sigma \quad \text{for } E\in\mathcal{E}_{h,\Omega}, \\
\pi_{0,E'}R_N &:= \frac{1}{\lceil E'\rceil}\int_{E'} R_N\,\mathrm{d}\sigma \quad \text{for } E'\in\mathcal{E}_{h,N}
\end{aligned} \tag{23}$$

onto the space of constant functions. An appropriate residual a posteriori error estimator is then given by

$$\begin{aligned}
\eta_{R,T} := \Big\{h_{\theta,T}^2\lceil\!\lceil \pi_{0,T}R_T\rceil\!\rceil_{0,T}^2 &+ \frac12\sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_{h,\Omega}} h_{\theta,T}\lceil\!\lceil \pi_{0,E}R_E\rceil\!\rceil_{0,E}^2 \\
&+ \sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_{h,N}} h_{\theta,T}\lceil\!\lceil \pi_{0,E}R_N\rceil\!\rceil_{0,E}^2\Big\}^{1/2},
\end{aligned} \tag{24}$$

From Relation (22) and the triangle inequality, we derive an upper bound for the error, which is formulated in the following theorem.

Theorem 6.1 (Upper error bound) *Let u and u_h be the exact solutions of Problems (3) and (12) and let $\mathcal{T}_h$ be an isotropic triangulation of a regular spherical domain Ω. Then the following error estimate holds:*

$$
\begin{aligned}
\lceil u - u_h \rceil_{1,G} \lesssim \Big\{ \sum_{T \in \mathcal{T}_h} \eta_{R,T}^2 \quad & + \sum_{T \in \mathcal{T}_h} h_{\theta,T}^2 \|\lceil R_T - \pi_{0,T} R_T \rceil\|_{0,T}^2 \\
& + \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_{h,\Omega}} h_{\theta,T} \|\lceil R_E - \pi_{0,E} R_E \rceil\|_{0,E}^2 \\
& + \sum_{E \in \mathcal{E}_{h,N}} h_{\theta,T} \|\lceil R_N - \pi_{0,E} R_N \rceil\|_{0,E}^2 \Big\}^{1/2} .
\end{aligned} \tag{25}
$$

6.2 Efficiency

The aim of this subsection is to show the validity of an estimate in converse direction which is similar to (25). To this end, we proceed as suggested by Verfürth [17] and consider bubble functions. Let $\phi_{x_{1,T}}$, $\phi_{x_{2,T}}$, $\phi_{x_{3,T}}$ be the nodal basis functions of $T \in \mathcal{T}_h$, see Remark 5.1. Due to Remark 5.2, the corresponding *triangle-bubble* function is given by

$$
\mathfrak{b}_T := \begin{cases} 27 \phi_{x_{1,T}} \phi_{x_{2,T}} \phi_{x_{2,T}} & \text{on } T \\ 0 & \text{on } G \setminus T. \end{cases}
$$

For a given edge $E \in \mathcal{E}_{h,\Omega}$ with $\omega_E = T_1 \cup T_2$, we enumerate the vertices of T_1 and T_2 such that the vertices of E are counted first and define the corresponding *edge-bubble* function by

$$
\mathfrak{b}_E = \begin{cases} 4 \phi_{x_{1,T_i}} \phi_{x_{2,T_i}} & \text{on } T_i,\ i = 1, 2 \\ 0 & \text{on } G \setminus \omega_E. \end{cases}
$$

This definition can easily be extended to edges on the boundary. Standard scaling arguments and Relations (4), (5), (9) and (10) prove the following Lemma.

Lemma 6.2 *Let $T \in \mathcal{T}_h$ and $E \in \mathcal{E}_h$ be arbitrary. Then the functions $\mathfrak{b}_T$ and $\mathfrak{b}_E$ have the subsequent properties:*

$$
\begin{array}{lll}
\operatorname{supp} \mathfrak{b}_T \subseteq T, & 0 \le \mathfrak{b}_T(\varphi, \theta) \le 1 \quad \forall (\varphi, \theta) \in T, & \max\limits_{(\varphi,\theta) \in T} \mathfrak{b}_T(\varphi, \theta) = 1 \\
\operatorname{supp} \mathfrak{b}_E \subseteq \omega_E, & 0 \le \mathfrak{b}_E(\varphi, \theta) \le 1 \quad \forall (\varphi, \theta) \in \omega_E, & \max\limits_{(\varphi,\theta) \in E} \mathfrak{b}_T(\varphi, \theta) = 1,
\end{array}
$$

$$
\begin{array}{lllll}
\int\limits_T \mathfrak{b}_T \, \mathrm{d}\omega & \sim & \lceil T \rceil & \sim & \int\limits_T \mathfrak{b}_T^2 \, \mathrm{d}\omega, \\
\int\limits_E \mathfrak{b}_E \, \mathrm{d}\sigma & \sim & \lceil E \rceil & \sim & \int\limits_E \mathfrak{b}_E^2 \, \mathrm{d}\sigma, \\
\int\limits_{T'} \mathfrak{b}_E \, \mathrm{d}\omega & \sim & \lceil T' \rceil & \sim & \int\limits_{T'} \mathfrak{b}_E^2 \, \mathrm{d}\omega \quad \textit{for all } T' \subset \omega_E,
\end{array}
$$

$$
\begin{array}{lll}
\|\lceil \nabla_{\mathcal{S}} \mathfrak{b}_T \rceil\|_{0,T} & \lesssim & h_{\theta,T}^{-1} \|\lceil \mathfrak{b}_T \rceil\|_{0,T} \\
\|\lceil \nabla_{\mathcal{S}} \mathfrak{b}_E \rceil\|_{0,T'} & \lesssim & h_{\theta,T'}^{-1} \|\lceil \mathfrak{b}_E \rceil\|_{0,T'} \quad \textit{for all } T' \subset \omega_E.
\end{array}
$$

A variant of Lemma 6.2 for plane domains was formulated, for example, by Verfürth [17]. Some ideas of its proof were outlined, for instance, by Kunert [10]. Note, that further arguments, like Condition (9), are necessary to verify these relations for spherical domains.

Now let $T \in \mathcal{T}_h$ and define $w_T := (\pi_{0,T} R_T)\mathfrak{b}_T$. This function vanishes on ∂T; we especially have $\operatorname{supp} w_T \subseteq \overline{T}$ and $\nabla_{\mathcal{S}} w \equiv 0$ on $G \setminus \overline{T}$. Equation (20) provides

$$\int_T R_T w_T \,\mathrm{d}\omega = \int_T \nabla_{\mathcal{S}}(u - u_h) \cdot \nabla_{\mathcal{S}} w_T \,\mathrm{d}\omega. \tag{26}$$

Since $\pi_{0,T} R_T$ is constant over T with respect to φ and θ, Lemma 6.2 implies

$$\int_T \pi_{0,T} R_T w_T \,\mathrm{d}\omega \sim \lceil T \rceil |\pi_{0,T} R_T|^2 = \|\lceil \pi_{0,T} R_T \rceil\|_{0,T}^2. \tag{27}$$

From Equations (26), (27) and Lemma 6.2, we obtain moreover

$$\begin{aligned}
\|\lceil \pi_{0,T} R_T \rceil\|_{0,T}^2 &\sim \int_T \pi_{0,T} R_T w_T \,\mathrm{d}\omega \;=\; \int_T R_T w_T \,\mathrm{d}\omega + \int_T (\pi_{0,T} R_T - R_T) w_T \,\mathrm{d}\omega \\
&= \int_T \nabla_{\mathcal{S}}(u - u_h) \cdot \nabla_{\mathcal{S}} w_T \,\mathrm{d}\omega + \int_T (\pi_{0,T} R_T - R_T) w_T \,\mathrm{d}\omega \\
&\le \lceil u - u_h \rceil_{1,T} \|\lceil \nabla_{\mathcal{S}} w_T \rceil\|_{0,T} + \|\lceil R_T - \pi_{0,T} R_T \rceil\|_{0,T} \|\lceil w_T \rceil\|_{0,T} \\
&\lesssim \lceil u - u_h \rceil_{1,T} h_{\theta,T}^{-1} |\pi_{0,T} R_T| \|\lceil \mathfrak{b}_T \rceil\|_{0,T} + \|\lceil R_T - \pi_{0,T} R_T \rceil\|_{0,T} |\pi_{0,T} R_T| \|\lceil \mathfrak{b}_T \rceil\|_{0,T} \\
&= \|\lceil \pi_{0,T} R_T \rceil\|_{0,T} \Big\{ h_{\theta,T}^{-1} \lceil u - u_h \rceil_{1,T} + \|\lceil R_T - \pi_{0,T} R_T \rceil\|_{0,T} \Big\}.
\end{aligned}$$

Division by $\|\lceil \pi_{0,T} R_T \rceil\|_{0,T}$ yields

$$\|\lceil \pi_{0,T} R_T \rceil\|_{0,T} \lesssim h_{\theta,T}^{-1} \lceil u - u_h \rceil_{1,T} + \|\lceil R_T - \pi_{0,T} R_T \rceil\|_{0,T}. \tag{28}$$

Now let $E \in \mathcal{E}_{h,\Omega}$ and consider the function $w_E := \pi_{0,E} R_E \mathfrak{b}_E$. Obviously, $\operatorname{supp} w_E \subseteq \overline{\omega_E}$ and therefore $w_E = 0$ on $\partial \omega_E$ and $\nabla_{\mathcal{S}} w_E = 0$ on $G \setminus \omega_E$. Hence, Equation (20) implies

$$\int_E R_E w_E \,\mathrm{d}\sigma = \int_{\omega_E} \nabla_{\mathcal{S}}(u - u_h) \cdot \nabla_{\mathcal{S}} w_E \,\mathrm{d}\omega - \sum_{T' \subset \omega_E} \int_{T'} R_{T'} w_E \,\mathrm{d}\omega \tag{29}$$

From Lemma 6.2, one concludes furthermore

$$\int_E \pi_{0,E} R_E w_E \,\mathrm{d}\sigma \sim \lceil E \rceil |\pi_{0,E} R_E|^2 = \|\lceil \pi_{0,E} R_E \rceil\|_{0,E}^2. \tag{30}$$

Using Relation (10), one obtains from Equations (30), (29) and Lemma 6.2

$$\begin{aligned}
\|\lceil \pi_{0,E} R_E \rceil\|_{0,E}^2 &\sim \int_E \pi_{0,E} R_E w_E \,\mathrm{d}\sigma \;=\; \int_E R_E w_E \,\mathrm{d}\sigma + \int_E (\pi_{0,E} R_E - R_E) w_E \,\mathrm{d}\sigma \\
&\le \lceil u - u_h \rceil_{1,\omega_E} \|\lceil \nabla_{\mathcal{S}} w_E \rceil\|_{0,\omega_E} \\
&\quad + \sum_{T' \subset \omega_E} \|\lceil R_{T'} \rceil\|_{0,T'} \|\lceil w_E \rceil\|_{0,T'} + \|\lceil R_E - \pi_{0,E} R_E \rceil\|_{0,E} \|\lceil w_E \rceil\|_{0,E}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \lceil u-u_h\rceil_{1,\omega_E}\sum_{T'\subset\omega_E} h_{\theta,T'}^{-1}|\pi_{0,E}R_E|\|\lceil\mathfrak{b}_E\rceil\|_{0,T'}\\
&\quad+\sum_{T'\subset\omega_E}\|\lceil R_{T'}\rceil\|_{0,T'}|\pi_{0,E}R_E|\|\lceil\mathfrak{b}_E\rceil\|_{0,T'}+\|\lceil R_E-\pi_{0,E}R_E\rceil\|_{0,E}|\pi_{0,E}R_E|\|\lceil\mathfrak{b}_E\rceil\|_{0,E}\\
&\lesssim \|\lceil\pi_{0,E}R_E\rceil\|_{0,E}\Big\{\lceil E\rceil^{-1/2}\lceil u-u_h\rceil_{1,\omega_E}\\
&\qquad+\lceil E\rceil^{-1/2}\sum_{T'\subset\omega_E}h_{\theta,T'}\|\lceil R_{T'}\rceil\|_{0,T'}+\|\lceil R_E-\pi_{0,E}R_E\rceil\|_{0,E}\Big\}\\
&\lesssim \|\lceil\pi_{0,E}R_E\rceil\|_{0,E}\Big\{\sum_{T'\subset\omega_E}h_{\theta,T'}^{-1/2}\lceil u-u_h\rceil_{1,\omega_E}+\sum_{T'\subset\omega_E}h_{\theta,T'}^{1/2}\|\lceil R_{T'}-\pi_{0,T'}R_{T'}\rceil\|_{0,T'}\\
&\qquad+\sum_{T'\subset\omega_E}h_{\theta,T'}^{1/2}\|\lceil\pi_{0,T'}R_{T'}\rceil\|_{0,T'}+\|\lceil R_E-\pi_{0,E}R_E\rceil\|_{0,E}\Big\}.
\end{aligned}
$$

Insertion of Inequality (28) and division by $\|\lceil\pi_{0,E}R_E\rceil\|_{0,E}$ yield

$$
\begin{aligned}
\|\lceil\pi_{0,E}R_E\rceil\|_{0,E} &\lesssim \sum_{T'\subset\omega_E}h_{\theta,T'}\lceil u-u_h\rceil_{1,\omega_E}\\
&\quad+\sum_{T'\subset\omega_E}h_{\theta,T'}^{1/2}\|\lceil R_{T'}-\pi_{0,T'}R_{T'}\rceil\|_{0,T'}+\|\lceil R_E-\pi_{0,E}R_E\rceil\|_{0,E}.
\end{aligned}\tag{31}
$$

Finally, for edges on the Neumann boundary $E\in\mathcal{E}_{h,N}$, we define $w_E:=\pi_{0,E}R_N\mathfrak{b}_E$. Lemma 6.2 provides

$$
\int_E \pi_{0,E}R_Nw_E\,\mathrm{d}\sigma\sim\lceil E\rceil|\pi_{0,E}R_N|^2=\|\lceil\pi_{0,E}R_N\rceil\|_{0,E}^2.\tag{32}
$$

With the same arguments as above, we obtain

$$
\begin{aligned}
\|\lceil\pi_{0,E}R_N\rceil\|_{0,E}^2 &\sim \int_E\pi_{0,E}R_Nw_E\,\mathrm{d}\sigma=\int_E R_Nw_E\,\mathrm{d}\sigma+\int_E(\pi_{0,E}R_N-R_N)w_E\,\mathrm{d}\sigma\\
&= \int_{\omega_E}\nabla_{\mathcal{S}}(u-u_h)\cdot\nabla w_E\,\mathrm{d}\omega-\int_{\omega_E}R_{T'}w_E\,\mathrm{d}\omega+\int_E(\pi_{0,E}R_N-R_N)w_E\,\mathrm{d}\sigma\\
&\lesssim \lceil u-u_h\rceil_{1,\omega_E}|\pi_{0,E}R_N|\|\lceil\nabla_{\mathcal{S}}\mathfrak{b}_E\rceil\|_{0,T'}\\
&\quad+\|\lceil R_{T'}\rceil\|_{0,T'}|\pi_{0,E}R_N|\|\lceil\mathfrak{b}_E\rceil\|_{0,T'}+\|\lceil R_N-\pi_{0,E}R_N\rceil\|_{0,E}|\pi_{0,E}R_N|\|\lceil\mathfrak{b}_E\rceil\|_{0,E}\\
&\lesssim \|\lceil\pi_{0,E}R_N\rceil\|_{0,E}\Big\{\lceil E\rceil^{-1/2}\lceil u-u_h\rceil_{1,\omega_E}\\
&\qquad+\lceil E\rceil^{-1/2}h_{\theta,T'}\|\lceil R_{T'}\rceil\|_{0,T'}+\|\lceil R_N-\pi_{0,E}R_N\rceil\|_{0,E}\Big\}\\
&\lesssim \|\lceil\pi_{0,E}R_N\rceil\|_{0,E}\Big\{h_{\theta,T'}^{-1/2}\lceil u-u_h\rceil_{1,\omega_E}\\
&\qquad+h_{\theta,T'}^{1/2}\|\lceil R_{T'}-\pi_{0,T'}R_{T'}\rceil\|_{0,T'}+\|\lceil R_N-\pi_{0,E}R_N\rceil\|_{0,E}\Big\},
\end{aligned}
$$

where $T'=\omega_E$. Division by $\|\lceil\pi_{0,E}R_N\rceil\|_{0,E}$ yields

$$
\|\lceil\pi_{0,E}R_N\rceil\|_{0,E}\lesssim h_{\theta,T'}^{-1/2}\lceil u-u_h\rceil_{1,\omega_E}+h_{\theta,T'}^{1/2}\|\lceil R_{T'}-\pi_{0,T'}R_{T'}\rceil\|_{0,T'}+\|\lceil R_N-\pi_{0,E}R_N\rceil\|_{0,E}.\tag{33}
$$

Collecting the estimates (11), (25), (28), (31) and (33), the following theorem is proven.

Theorem 6.3 (Lower error bound) *Let u and u_h be the exact solutions of Problems (3) and (12) and let $\mathcal{T}_h$ be an isotropic triangulation of a regular spherical domain Ω. Then the estimate*

$$\begin{aligned}\eta_{R,T} \lesssim \Big\{ \lceil u - u_h \rceil^2_{1,\omega_T} \;&+\; \sum_{T' \subset \omega_T} h^2_{\theta,T'} \| \lceil R_{T'} - \pi_{0,T'} R_{T'} \rceil\|^2_{0,T'} \\ &+\; h_{\theta,T} \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_{h,\Omega}} \| \lceil R_E - \pi_{0,E} R_E \rceil\|^2_{0,E} \\ &+\; h_{\theta,T} \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_{h,N}} \| \lceil R_N - \pi_{0,E} R_N \rceil\|^2_{0,E} \Big\}^{1/2}\end{aligned}$$

holds for all $T \in \mathcal{T}_h$, where R_T, R_E, R_N, $\pi_{0,T} R_T$, $\pi_{0,E} R_E$, $\pi_{0,E} R_N$ and $\eta_{R,T}$ are given by (21), (23) and (24).

7 Conclusion

In this paper, a mixed boundary value problem was considered for the Laplace-Beltrami operator on a subdomain of the three-dimensional unit sphere $\mathcal{S}^2$. We discussed a conforming first-order finite element discretization and derived an a posteriori error estimator that provides a computable global upper bound and a local lower bound for the error of the solution of the given boundary value problem. The general framework of the theory is partly similar to the standard methodology, see Verfürth [17]; the ingredients are carefully adapted to suit our needs.

Basic estimates in usual Sobolev spaces were generalized to spherical domains. The usage of spherical coordinates simplifies the consideration of spherical domains, but yields weighted norms and singularities in the parametrization at the poles.

An interpolation operator of Clément-type (see for example Clément [6]) was introduced such that it operates on functions that are defined on the unit sphere. The local estimates obtained for this operator are pretty similar to the "usual" interpolation error estimates. For the simplification of the proofs in the deduction of the error estimates, we restricted ourselves to regular domains.

The theory of interpolation error estimates on the sphere finds further application, for instance, in linear elasticity, cf. [3]. For example, the solution of the Lamé problem in the neighbourhood of corner singularities leads to a second-order eigenvalue problem on spherical domains. The extension of the current results to these problems is a topic of future work.

A Proof of Lemma 4.1

Lemma 4.1 (Trace theorem for spherical domains) Let $T \in \mathcal{T}_h$ and $E \in \mathcal{E}(T)$. Then all $v \in \mathcal{H}^1(T)$ satisfy

$$\| \lceil v \rceil\|^2_{0,E} \lesssim \frac{\lceil E \rceil}{\lceil T \rceil} \Big(\| \lceil v \rceil\|^2_{0,T} + h^2_{\theta,T} \lceil v \rceil^2_{1,T} \Big).$$

Proof The idea of the proof is to transform T to a fixed reference element $\hat{T}$ with a reference edge $\hat{E}$ corresponding to E. Accordingly, denote by $\hat{v}$ the transformed function v. Then we can exploit the trace theorem

$$\|\hat{v}\|_{0,\hat{E}} \leq c\|\hat{v}\|_{1,\hat{T}}, \tag{34}$$

where the constant c is independent of v and T. We distinguish two cases.

Suppose first $\vartheta_{-,T} > 0$. There is an affine linear map $B : \hat{T} \mapsto T$ that maps $\hat{T}$ to T and $\hat{E} \subset \partial\hat{T}$ to E, i.e.

$$(\varphi, \theta)^\top = B(\hat{x}, \hat{y})^\top + b$$

with a displacement vector b. The gradients of $\hat{v}$ and v are related by

$$\hat{\nabla}\hat{v} = B^\top \nabla v = B^\top D \nabla_{\mathcal{S}} v\,, \quad \text{where } D = \operatorname{diag}(\sin\theta, 1).$$

Obviously $|\hat{\nabla}\hat{v}| = |B^\top D \nabla_{\mathcal{S}} v| \leq \|B^\top D\| |\nabla_{\mathcal{S}} v|$. All matrix norms are equivalent, i.e. we can choose the spectral norm which satisfies

$$\|B^\top D\|_2 = \|DB\|_2 = \sup_{\mathbf{x}\in \mathbb{R}^2\setminus\{0\}} \frac{\|DB\mathbf{x}\|_2}{\|\mathbf{x}\|_2} = \sup_{\mathbf{x}:\|x\|_2=\rho_{\hat{T}}} \frac{\|DB\mathbf{x}\|_2}{\rho_{\hat{T}}},$$

where $\rho_{\hat{T}}$ denotes the diameter of the largest ball inscribed to $\hat{T}$. Consider any vector $\mathbf{x}$ in the largest ball inscribed to $\hat{T}$ with length $\|\mathbf{x}\|_2 = \rho_{\hat{T}}$. While $\hat{T}$ is mapped by B to a triangle T^0 in the parameter plane which corresponds to T moved to the origin, the vector $\mathbf{x}$ is mapped to some vector in the inside of T^0.

Further, the application of D compresses T^0 in φ-direction by the factor $\sin\theta$. Due to Assumption (5), the corresponding triangle $\tilde{T} = (DB)(\hat{T})$ is isotropic, i.e. its dimensions in φ- and θ-directions are both equivalent to $h_{\theta,T}$. Thus, the vector $DB\mathbf{x}$ has the length $\|DB\mathbf{x}\|_2 \leq \operatorname{diam}(\tilde{T}) \lesssim h_{\theta,T}$. Since $\rho_{\hat{T}} \sim 1$, this gives $\|B^\top D\|_2 \lesssim h_{\theta,T}$ and thus

$$|\hat{\nabla}\hat{v}| \lesssim h_{\theta,T} |\nabla_{\mathcal{S}} v|. \tag{35}$$

Moreover, it holds that $\|\hat{v}\|^2_{0,\hat{E}} = |\hat{E}|/|E| \cdot \|v\|^2_{0,E}$ and $\|\hat{v}\|^2_{0,\hat{T}} = |\hat{T}|/|T| \cdot \|v\|^2_{0,T}$. Edges are given in parametrized form $(\varphi(t), \theta(t))$. The corresponding derivatives $\dot{\varphi}$ and $\dot{\theta}$ are constant with respect to t. Relation (9) yields

$$\begin{aligned}
\frac{\|\lceil v\rceil\|^2_{0,E}}{\|v\|^2_{0,E}} &= \frac{\int_0^1 v^2 \sqrt{\dot{\varphi}^2 \sin^2\theta(t) + \dot{\theta}^2}\,\mathrm{dt}}{\int_0^1 v^2 \sqrt{\dot{\varphi}^2 + \dot{\theta}^2}\,\mathrm{dt}} \sim \frac{\sqrt{\dot{\varphi}^2 \frac{h^2_{\theta,T}}{h^2_{\varphi,T}} + \dot{\theta}^2} \int_0^1 v^2\,\mathrm{dt}}{\sqrt{\dot{\varphi}^2 + \dot{\theta}^2} \int_0^1 v^2\,\mathrm{dt}} \sim \frac{\lceil E\rceil}{|E|}, \\
\|v\|^2_{0,T} &\sim \frac{h_{\varphi,T}}{h_{\theta,T}} \|\lceil v\rceil\|^2_{0,T}, \qquad \Longrightarrow \qquad |T| = \|1\|^2_{0,T} \sim \frac{h_{\varphi,T}}{h_{\theta,T}} \|\lceil 1\rceil\|^2_{0,T} = \frac{h_{\varphi,T}}{h_{\theta,T}} \lceil T\rceil.
\end{aligned}$$

Insertion into Inequality (34) implies

$$\begin{aligned}
\|\lceil v\rceil\|^2_{0,E} &= \frac{\lceil E\rceil}{|E|} \|v\|^2_{0,E} = \frac{\lceil E\rceil}{|\hat{E}|} \|\hat{v}\|^2_{0,\hat{E}} \lesssim \frac{\lceil E\rceil}{|\hat{E}|} \|\hat{v}\|^2_{1,\hat{T}} = \frac{\lceil E\rceil}{|\hat{E}|} \Big(\|\hat{v}\|^2_{0,\hat{T}} + \|\hat{\nabla}\hat{v}\|^2_{0,\hat{T}}\Big) \\
&\lesssim \frac{\lceil E\rceil}{|\hat{E}|} \Big(\frac{|\hat{T}|}{|T|} \|v\|^2_{0,T} + h^2_{\theta,T} \frac{|\hat{T}|}{|T|} \|\nabla_{\mathcal{S}} v\|^2_{0,T}\Big) \sim \frac{\lceil E\rceil}{\lceil T\rceil} \Big(\|\lceil v\rceil\|^2_{0,T} + h^2_{\theta,T} \lceil v\rceil^2_{1,T}\Big).
\end{aligned}$$

Now let $T \in \mathcal{T}_h$ with $\vartheta_{-,T} = 0$. Let the corresponding reference domain $\hat{T}$ be given by the image of the transformation

$$\hat{x} = \vartheta_{+,T}^{-1} \cos\varphi \sin\theta, \qquad \hat{y} = \vartheta_{+,T}^{-1} \sin\varphi \sin\theta.$$

This is a sector of the circle $\{\hat{x}^2 + \hat{y}^2 \leq 1\}$ with angle $h_{\varphi,T} \sim 1$, see Relation (6). Due to the assumptions on the mesh, we have $|\cos\theta| \sim 1$ for all $(\varphi,\theta) \in T$. The Jacobian of the transformation then reads $|J| = \vartheta_{+,T}^{-2} |\sin\theta\cos\theta| \sim \vartheta_{+,T}^{-2} \sin\theta$.

Moreover, one obtains

$$\begin{aligned} \frac{\partial v}{\partial\varphi} &= \vartheta_{+,T}^{-1}\left[-\frac{\partial\hat{v}}{\partial\hat{x}} \sin\varphi\sin\theta + \frac{\partial\hat{v}}{\partial\hat{y}} \cos\varphi\sin\theta \right], \\ \frac{\partial v}{\partial\theta} &= \vartheta_{+,T}^{-1}\left[\frac{\partial\hat{v}}{\partial\hat{x}} \cos\varphi\cos\theta + \frac{\partial\hat{v}}{\partial\hat{y}} \sin\varphi\cos\theta \right] \end{aligned}$$

and thus

$$\Big(\frac{\vartheta_{+,T}}{\sin\theta}\frac{\partial v}{\partial\varphi}\Big)^2 + \Big(\frac{\vartheta_{+,T}}{\cos\theta}\frac{\partial v}{\partial\theta}\Big)^2 = \Big(\frac{\partial\hat{v}}{\partial\hat{x}}\Big)^2 + \Big(\frac{\partial\hat{v}}{\partial\hat{y}}\Big)^2.$$

For the vertical edges, we have $\varphi(t) = const$, $\theta(t) = t\,h_{\theta,T}$ (or $\theta(t) = \pi - t\,h_{\theta,T}$), i.e. $\lceil E \rceil = h_{\theta,T}$, and thus

$$\lceil\!\lceil v \rceil\!\rceil_{0,E}^2 = \int_0^1 v^2 \sqrt{h_{\theta,T}^2}\,\mathrm{d}t = \lceil E \rceil / |\hat{E}| \cdot \|\hat{v}\|_{0,\hat{E}}^2.$$

The horizontal edge at $\theta \equiv h_{\theta,T}$ (or $\theta \equiv \pi - h_{\theta,T}$) with $\varphi(t) = t\,h_{\varphi,T}$ satisfies $\lceil E \rceil = h_{\varphi,T}\vartheta_{+,T}$ per definition (recall $\vartheta_{+,T} = \sin\theta = \sin h_{\theta,T}$) and

$$\lceil\!\lceil v \rceil\!\rceil_{0,E}^2 = \int_0^1 v^2 \sqrt{h_{\varphi,T}^2 \sin^2 h_{\theta,T}}\,\mathrm{d}t = \lceil E \rceil / |\hat{E}| \cdot \|\hat{v}\|_{0,\hat{E}}^2.$$

Since $|\hat{E}| \sim 1$ for all corresponding reference edges, we get from Relation (10)

$$\lceil\!\lceil v \rceil\!\rceil_{0,E}^2 \sim \lceil E \rceil \|\hat{v}\|_{0,\hat{E}}^2 \sim h_{\theta,T} \|\hat{v}\|_{0,\hat{E}}^2$$

in both cases. The relations $\mathrm{d}\hat{x}\mathrm{d}\hat{y} = |J|\,\mathrm{d}\varphi\,\mathrm{d}\theta \sim \vartheta_{+,T}^{-2}\,\mathrm{d}\omega$, $\vartheta_{+,T} = \sin h_{\theta,T} \sim h_{\theta,T}$ and $|\cos\theta| \sim 1$ yield

$$\begin{aligned} \|\hat{v}\|_{0,\hat{T}}^2 &= \int_{\hat{T}} \hat{v}^2\,\mathrm{d}\hat{x}\mathrm{d}\hat{y} = \int_T v^2 |J|\,\mathrm{d}\varphi\,\mathrm{d}\theta \sim \vartheta_{+,T}^{-2} \int_T v^2\,\mathrm{d}\omega = h_{\theta,T}^{-2} \lceil\!\lceil v \rceil\!\rceil_{0,T}^2 \\ |\hat{v}|_{1,\hat{T}}^2 &= \int_{\hat{T}} \Big\{ \Big(\frac{\partial\hat{v}}{\partial\hat{x}}\Big)^2 + \Big(\frac{\partial\hat{v}}{\partial\hat{y}}\Big)^2 \Big\}\,\mathrm{d}\hat{x}\mathrm{d}\hat{y} = \vartheta_{+,T}^{-2} \int_T \Big\{ \Big(\frac{\vartheta_{+,T}}{\sin\theta}\frac{\partial v}{\partial\varphi}\Big)^2 + \Big(\frac{\vartheta_{+,T}}{\cos\theta}\frac{\partial v}{\partial\theta}\Big)^2 \Big\}\,\mathrm{d}\omega \\ &\sim \vartheta_{+,T}^{-2} \int_T \Big\{ \Big(\frac{1}{\sin\theta}\frac{\partial v}{\partial\varphi}\Big)^2 + \Big(\frac{\partial v}{\partial\theta}\Big)^2 \Big\} \vartheta_{+,T}^2\,\mathrm{d}\omega \sim \lceil v \rceil_{1,T}^2 \end{aligned}$$

From Relation (34), we finally obtain

$$\begin{aligned} \lceil\!\lceil v \rceil\!\rceil_{0,E}^2 &\sim h_{\theta,T} \|\hat{v}\|_{0,\hat{E}}^2 \lesssim h_{\theta,T} (\|\hat{v}\|_{0,\hat{T}}^2 + |\hat{v}|_{1,\hat{T}}^2) \sim h_{\theta,T} (h_{\theta,T}^{-2} \lceil\!\lceil v \rceil\!\rceil_{0,T}^2 + \lceil v \rceil_{1,T}^2) \\ &\sim h_{\theta,T}^{-1} (\lceil\!\lceil v \rceil\!\rceil_{0,T}^2 + h_{\theta,T}^2 \lceil v \rceil_{1,T}^2). \end{aligned}$$

With (10), the Lemma is proven also for the second case. ■

B Proof of Lemma 4.2

Lemma 4.2 (Bramble-Hilbert lemma for spherical domains) Let $T \in \mathcal{T}_h$, $E \in \mathcal{E}(T)$, $x \in \mathcal{N}(T)$ and $\omega = T$, $\omega = \omega_x$, $\omega = \tilde{\omega}_T$ or $\omega = \tilde{\omega}_E$. For all functions $v \in \mathcal{H}^1(\omega)$ there is a constant function $\tau = \tau(v) \in \mathcal{P}_{0|\omega}$, such that

$$|\!\lceil v - \tau \rceil\!|_{0,\omega} \lesssim h_{\theta,T} \lceil v \rceil_{1,\omega}.$$

Proof It was proven by Dupont, Scott [8], see also Brenner, Scott [4], that the estimate

$$\forall \hat{v} \in H^1(\hat{\omega}) \; \exists \hat{\tau} \in \mathcal{P}_0 : \qquad \|\hat{v} - \hat{\tau}\|_{0,\hat{\omega}} \leq c_\tau(\hat{\omega}) |\hat{v}|_{1,\hat{\omega}}, \tag{36}$$

holds for any domain $\hat{\omega} \subset I\!R^2$ which is star-shaped with respect to a ball $\mathcal{B}$. Then the constant $c_\tau(\hat{\omega})$ depends on the ratio of the diameter of $\hat{\omega}$ and the radius of the largest ball with respect to which $\hat{\omega}$ is star-shaped. In order to exploit Inequality (36), we will transform ω to a reference domain $\hat{\omega}$ which is star-shaped with respect to a ball of fixed size.

In the neighbourhood of the poles, i.e. if $\vartheta_{-,\omega} = 0$, there is only a limited number of possible structures of ω. Due to the assumptions on the mesh (especially no crack tips in the pole elements), there is at least one element $T' \subset \omega$, such that ω is star-shaped with respect to the largest ball inscribed to T'.

We *flatten* the polar cap using

$$\hat{x} = \vartheta_{+,\omega}^{-1} \cos \wp \frac{\varphi - \varphi_0}{h_{\varphi,\omega}} \sin\theta, \qquad \hat{y} = \vartheta_{+,\omega}^{-1} \sin \wp \frac{\varphi - \varphi_0}{h_{\varphi,\omega}} \sin\theta,$$

where $\varphi_0 = \inf_{(\varphi,\theta)\in\omega} \varphi$, such that ω is transformed to a part of a fixed sector $\hat{\omega}$ with opening angle $\wp = \frac{\pi}{2}$ of the unit circle $\{\hat{x}^2 + \hat{y}^2 \leq 1\}$. Then, $\hat{\omega}$ is also star-shaped with respect to a ball, at least with respect to the largest ball inscribed to $\hat{T}'$ (the element corresponding to T'). The number of elements in ω is limited. Therefore all elements in $\hat{\omega}$ have the size $\mathcal{O}(1)$. Especially the radius $\varrho_{T'}$ of the largest ball inscribed to $\hat{T}'$ satisfies $\varrho_{T'} \sim 1$. Moreover, it follows from $\operatorname{diam}\hat{\omega} \sim 1$ that Inequality (36) hold with a constant $c_\tau \lesssim 1$, which is independent of ω and $\hat{\omega}$.

The Jacobian of the given transformation reads $|J| = \wp (h_{\varphi,\omega} \vartheta_{+,\omega}^2)^{-1} \sin\theta\cos\theta$. The partial derivatives are related by

$$\left[\left(\frac{\wp^{-1} h_{\varphi,\omega} \vartheta_{+,\omega}}{\sin\theta} \frac{\partial v}{\partial\varphi} \right)^2 + \left(\frac{\vartheta_{+,\omega}}{\cos\theta} \frac{\partial v}{\partial\theta} \right)^2 \right] = \left(\frac{\partial\hat{v}}{\partial\hat{x}} \right)^2 + \left(\frac{\partial\hat{v}}{\partial\hat{y}} \right)^2.$$

We required $h_{\theta,\omega} \leq \frac{\pi}{4}$ at the poles. Thus we have $|\cos\theta| \sim 1$ and conclude

$$\|\hat{v}\|_{0,\hat{\omega}}^2 = \int\limits_{\hat{\omega}} \hat{v}^2 \,\mathrm{d}\hat{x}\mathrm{d}\hat{y} = \int\limits_{\omega} |J| v^2 \,\mathrm{d}\varphi\,\mathrm{d}\theta \sim \wp (h_{\varphi,\omega}\vartheta_{+,\omega}^2)^{-1} |\!\lceil v \rceil\!|_{0,\omega}^2$$

and

$$\begin{aligned}
|\hat{v}|_{1,\hat{\omega}}^2 &= \int\limits_{\hat{\omega}} \left(\frac{\partial\hat{v}}{\partial\hat{x}} \right)^2 + \left(\frac{\partial\hat{v}}{\partial\hat{y}} \right)^2 \mathrm{d}\hat{x}\mathrm{d}\hat{y} \sim \int\limits_{\omega} |J| \left[\left(\frac{\wp^{-1} h_{\varphi,\omega}\vartheta_{+,\omega}}{\sin\theta} \frac{\partial v}{\partial\varphi} \right)^2 + \left(\frac{h_{\theta,\omega}}{\cos\theta} \frac{\partial v}{\partial\theta} \right)^2 \right] \mathrm{d}\varphi\,\mathrm{d}\theta \\
&\sim \wp (h_{\varphi,\omega}\vartheta_{+,\omega}^2)^{-1} \left(h_{\varphi,\omega}^2 \vartheta_{+,\omega}^2 |\!\lceil \sin^{-1}\theta \frac{\partial v}{\partial\varphi} \rceil\!|_{0,\omega}^2 + h_{\theta,\omega}^2 |\!\lceil \frac{\partial v}{\partial\theta} \rceil\!|_{0,\omega}^2 \right) \\
&\sim \wp (h_{\varphi,\omega}\vartheta_{+,\omega}^2)^{-1} h_{\theta,\omega}^2 \lceil v \rceil_{1,\omega}^2.
\end{aligned}$$

Insertion into (36) yields the assertion:

$$\lceil\!\lceil v-w \rceil\!\rceil_{0,\omega} \;\sim\; \wp^{-1/2}(h_{\varphi,\omega}\vartheta^2_{+,\omega})^{1/2}\|\hat{v}-\hat{w}\|_{0,\hat{\omega}} \leq \wp^{-1/2}(h_{\varphi,\omega}\vartheta^2_{+,\omega})^{1/2}c_{\mathcal{T}}(\hat{\omega})|\hat{v}|_{1,\hat{\omega}} \lesssim h_{\theta,\omega}\lceil v \rceil_{1,\omega}.$$

Now suppose $\vartheta_{-,\omega} > 0$, i.e. Relation (9) holds. Then we construct a reference domain $\hat{\omega}$ which is star-shaped with respect to the largest ball inscribed to $\hat{T}$ and a continuous, piecewise affine linear map which maps $\hat{\omega}$ to ω and $\hat{T}$ to T.

In the case $\omega = T$, we choose $\hat{\omega} = \hat{T} = \{(\hat{x},\hat{y}) \in I\!R^2 \mid 0 < \hat{x} < 1,\ 0 < \hat{y} < \hat{x}\}$.

If $\omega = \omega_x$ and $x \in \mathcal{N}_{h,\Omega}$, the domain ω can be transformed to a regular polygon with n_x nodes, where n_x denotes the number of elements in ω_x. If x is a boundary node, the reference domain $\hat{\omega}$ can be chosen as a quarter of the unit circle, divided into n_x equal triangles, see also Figure 4.

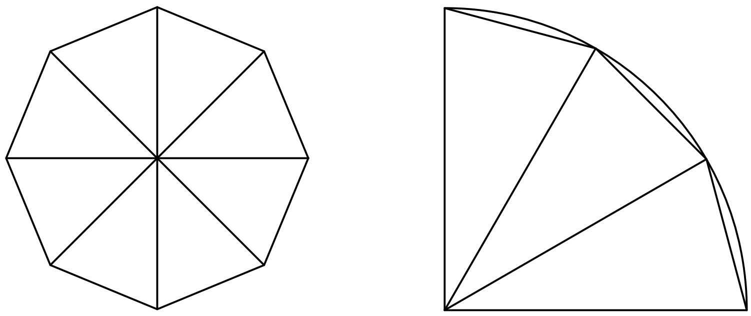
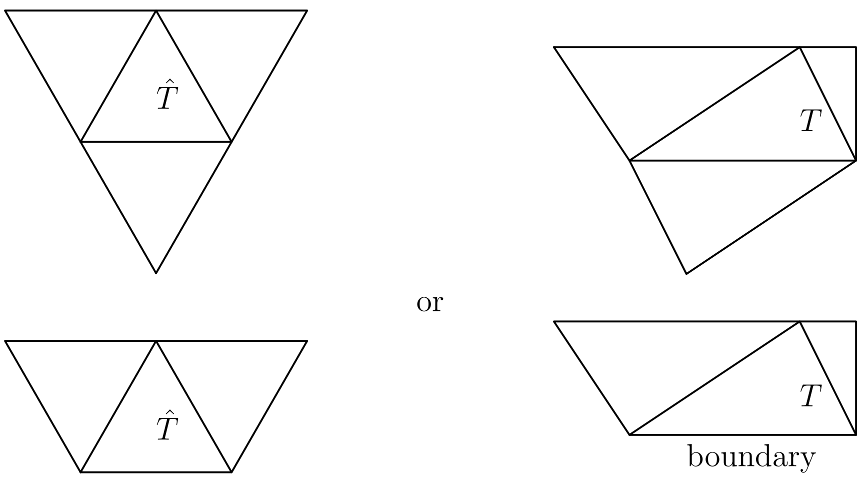


Figure 4: Reference domains for $\omega = \omega_x$; left hand side: $x \in \mathcal{N}_{h,G}$, right hand side: $x \in \mathcal{N}_{h,N} \cup \mathcal{N}_{h,D}$

The reference domain $\hat{\omega}$ corresponding to $\omega = \tilde{\omega}_T$ is described as follows:

- Transform T to the equilateral triangle $\hat{T}$ with the nodes $(0,0)$, $(1,0)$, $(\frac{1}{2}, \frac{\sqrt{3}}{2})$.

- For each element $T' \subset \omega_T \setminus T$ construct another equilateral triangle $\hat{T}'$ over the edge $\hat{E} \subset \hat{T}$ which corresponds to the edge $E \in \mathcal{E}_h$ with $T' \cup T = \omega_E$.

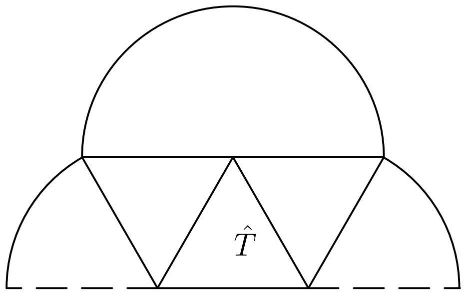


- For each node $\hat{x}$ of $\hat{T}$ (corresponding to $x \in \mathcal{N}(T)$) construct a circular arc with center $\hat{x}$, radius 1 and opening angle

  (a) $60°$, if $\omega_x \cap \omega_T$ contains exactly two triangles,

  (b) $180°$, if $\omega_x \cap \omega_T$ contains three triangles,

  as shown in the next figure.



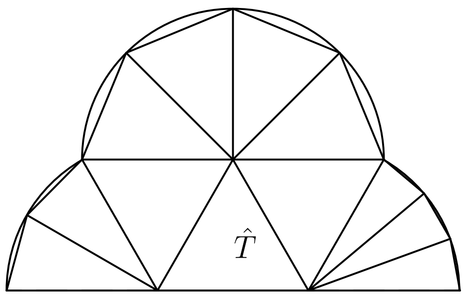
  If $\omega_x \cap \omega_T$ contains exactly one triangle, namely T, then no circular arc has to be constructed.

- Denote by n_x the number of elements in ω_x for each $x \in \mathcal{N}(T)$. Split the circular sector over $\hat{x}$ into

  (a) $n_x - 2$ or

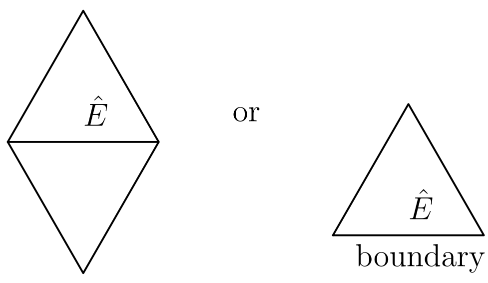
  (b) $n_x - 3$

  equal sectors and join the end points, see also the next figure.



If $\omega = \tilde{\boldsymbol{\omega}}_{\boldsymbol{E}}$ then $\hat{\omega}$ is described as follows:

- Let $\hat{E} = \{(\hat{x}, \hat{y}) \in \mathbb{R}^2 \mid 0 \leq \hat{x} \leq 1, \hat{y} = 0\}$ by the reference edge corresponding to E. For each $T' \subset \omega_E$ construct an equilateral triangle $\hat{T}'$ over $\hat{E}$.

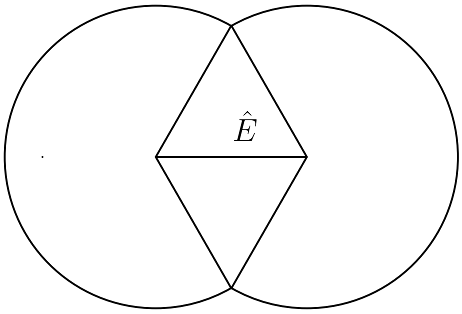


- For each node $\hat{x}$ of $\hat{E}$ corresponding to $x \in \mathcal{N}(E)$ construct a circular arc with middle point $\hat{x}$, radius 1 and angle

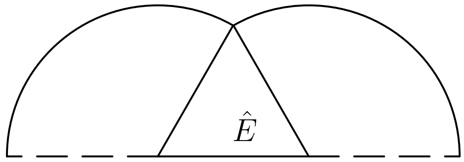
  (a) $120°$, if $\omega_x \cap \omega_E$ contains one element,

  (b) $240°$, if $\omega_x \cap \omega_E$ contains two elements,

  as shown in the next figure.



or

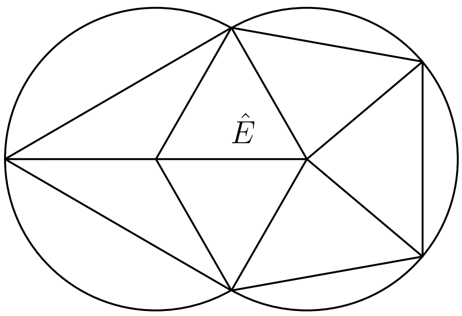


- Denote by n_x the number of elements in ω_x for each $x \in \mathcal{N}(E)$. Split the circular sector over $\hat{x}$ into

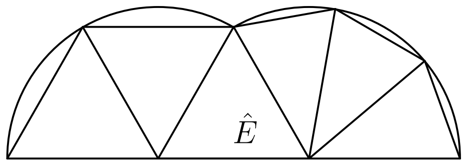
  (a) $n_x - 1$ or

  (b) $x_x - 2$

  equal sectors and join the end points, see also the next figure. This is possible, provided $n_x > 3$ for non-boundary nodes x. Otherwise rectangular triangles have to be chosen instead of equilateral ones.



or



Due to assumption (iii) the angles of all triangles are bounded. So is the number of elements in $\hat{\omega}$ or ω. Moreover, there is only a limited number of possible reference domains constructed in the described way. Each of these domains is star-shaped with respect to the largest ball inscribed to the triangle $\hat{T}$ which corresponds to T.

By construction, the radius of the largest ball inscribed to $\hat{T}$ equals $\frac{1}{2\sqrt{3}}$. The diameter of $\hat{\omega}$ is also $\mathcal{O}(1)$. So is the radius of the largest ball inscribed to $\hat{\omega}$. Hence the constant in (36) is independent of ω and the triangulation.

There is a continuous, piecewise affine linear map $F : \hat{\omega} \mapsto \omega$ which transforms $\hat{\omega}$ to ω. Enumerating the elements in ω from 1 to the z (where z is the number of elements in ω), the restriction $F_i := F|_{T_i}$ is affine linear and maps $\hat{T}_i$ to T_i. This means, we can write

$$(\varphi, \theta)^\top = F_i(\hat{x}, \hat{y}) = B_i(\hat{x}, \hat{y})^\top + b_i$$

with transformation matrices B_i and displacement vectors b_i, $i = 1, \ldots, z$.

From Estimate (35), we know that the gradients for each element T_i, $i = 1, \dots, z$, are related by

$$\left|\hat{\nabla}\hat{v}|_{\hat{T}_i}\right| \lesssim h_{\theta,T_i}\left|\nabla_{\mathcal{S}} v|_{T_i}\right|.$$

Relation (9) yields

$$\begin{aligned}
\|v\|_{0,\omega}^2 &= \int\limits_\omega v^2 \sin\theta \,\mathrm{d}\varphi\,\mathrm{d}\theta = \sum_{i=1}^{z} \int\limits_{T_i} v^2 \sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta \\
&\sim \sum_{i=1}^{z} \frac{h_{\theta,T_i}}{h_{\varphi,T_i}} \int\limits_{\hat{T}_i} \hat{v}^2 |\det B_i| \,\mathrm{d}\hat{x}\mathrm{d}\hat{y} \;=\; \sum_{i=1}^{z} \frac{h_{\theta,T_i}}{h_{\varphi,T_i}} |\det B_i| \|\hat{v}\|_{0,\hat{T}_i}^2
\end{aligned} \tag{37}$$

and

$$\begin{aligned}
|\hat{v}|_{1,\hat{\omega}}^2 &= \int\limits_{\hat{\omega}} |\hat{\nabla}\hat{v}|^2 \,\mathrm{d}\hat{x}\mathrm{d}\hat{y} = \sum_{i=1}^{z} \int\limits_{\hat{T}_i} |\hat{\nabla}\hat{v}|^2\,\mathrm{d}\hat{x}\mathrm{d}\hat{y} \\
&\lesssim \sum_{i=1}^{z} \int\limits_{T_i} h_{\theta,T_i} |\nabla_{\mathcal{S}} v|^2 |\det B_i|^{-1}\,\mathrm{d}\varphi\,\mathrm{d}\theta \\
&\sim \sum_{i=1}^{z} h_{\varphi,T_i} |\det B_i|^{-1} \int\limits_{T_i} |\nabla_{\mathcal{S}} v|^2 \sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta.
\end{aligned} \tag{38}$$

Since adjacent elements have approximately the same size, we have $h_{\varphi,T_i} \sim h_{\varphi,T}$, $h_{\theta,T_i} \sim h_{\theta,T}$, $|\det B_i| = |T_i|/|\hat{T}_i| \sim |T|$ for all i. We may conclude from (36), (37) and (38) that

$$\|v - \tau\|_{0,\omega}^2 \sim \frac{h_{\theta,T}}{h_{\varphi,T}} |T| \|\hat{v} - \hat{\tau}\|_{0,\hat{\omega}}^2 \lesssim \frac{h_{\theta,T}}{h_{\varphi,T}} |T| |\hat{v}|_{1,\hat{\omega}}^2 \lesssim h_{\theta,T}^2 |v|_{1,\omega}^2.$$

■

Remark The construction of the reference domains considered in the proof of Lemma 4.2 is based on work by Kunert [10].

Acknowledgement

We thank Gerd Kunert for helpful discussions as well as the DFG (Deutsche Forschungsgemeinschaft) for supporting this project (AP 72/1-1).

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