Finite element methods with anisotropic meshes near edges

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Abstract

We consider partial differential equations whose solutions have singular behaviour near edges. The idea is to reflect this in a finite element approximation by using *anisotropic meshes* with a small mesh size perpendicularly to the edge and a larger mesh size along the edge.

1 Motivation for using anisotropic meshes

Consider the Poisson equation with homogeneous Dirichlet boundary conditions in a prismatic domain $\Omega = G \times Z$, $G \subset I\!R^2$, $Z \subset I\!R^1$,

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1}$$

Assume that the two-dimensional polygonal base G has one corner with interior angle $\omega > \pi$ in the origin. Then Ω has a singular edge which is part of the x_3-axis. The solution u of (1) exhibits the well-known r^λ-singularity, where $r = r(x)$ is the distance of a point $x = (x_1, x_2, x_3)$ to the x_3-axis, and λ is determined by

$$\lambda = \frac{\pi}{\omega}.$$

Hence,

$$u \notin H^2(\Omega),$$

that means that the standard assumption for investigating the approximation order of a finite element method with piecewise linear basis functions is violated. But let us mention here that second order generalized derivatives of u exist in Ω. Only, some of them are not square integrable. However, when appropriately weighted, we get

$$r^\beta \frac{\partial u}{\partial x_1},\ r^\beta \frac{\partial u}{\partial x_2} \in H^1(\Omega) \quad \text{for } \beta > 1 - \lambda, \tag{2}$$

while for the derivative in edge direction even

$$\frac{\partial u}{\partial x_3} \in H^1(\Omega) \tag{3}$$

holds. For the proof of such results see [AN96, MR88].

Apply now the finite element method with piecewise linear basis functions to solve (1). On a sequence of quasi-uniform meshes $\mathcal{T}_h$ we obtain for any arbitrarily small positive number ε the error estimate

$$\|u - u_h\|_1 \le Ch^{\lambda-\varepsilon}\|f\|_0,$$

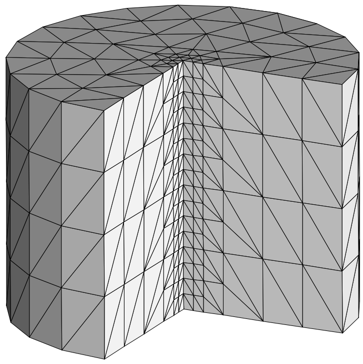

Figure 1: Isotropic mesh refinement near the edge.

where h is the maximum element diameter in $\mathcal{T}_h$. This result is almost sharp in the sense that there is an example problem (1) and a family $\{\mathcal{T}_h\}$ of meshes such that

$$\|u - u_h\|_1 \geq Ch^\lambda$$

with $C > 0$ [Ape91, page 85]. Consequently, locally refined meshes should be used.

If we use a standard adaptive finite element code, we end up with a mesh like that given in Figure 1. One can describe such meshes also a-priori by relating the diameter h_K of the elements K to the distance r_K of the center of gravity of K from the edge of Ω. Denote by μ and h the mesh grading parameter and the global mesh size. Then we require

$$h_K \sim h r_K^{1-\mu} \tag{4}$$

and we can prove for $\frac{1}{3} < \mu < \lambda$ the optimal convergence order,

$$\|u - u_h\|_1 \leq Ch\|f\|_0$$

[ASW96]. This method fails, however, if $\lambda \leq \frac{1}{3}$, which may occur in the case of mixed boundary conditions or piecewise smooth coefficients in the differential operator (interface problems). The reason for failing is that $\mu \leq \frac{1}{3}$ leads to a number of elements $N_e \sim h^{-3}|\ln h|$ for $\mu = \frac{1}{3}$ and $N_e \sim h^{1/\mu}$ for $\mu < \frac{1}{3}$, contrary to $N_e \sim h^{-3}$ for $\mu > \frac{1}{3}$.

On the other hand, a closer look at the structure of the solution reveals that the singular behaviour is restricted to the directions perpendicularly to the edge, see e. g. (2), while the solution is more regular in edge direction, see (3). This leads to the idea to refine the mesh only in the singular directions and to use the global mesh size h in x_3-direction.

Let $\mathcal{T}_h^{2d}$ be a finite element mesh of G which is refined according to (4). By using a quasi-uniform subdivision $\mathcal{T}_h^{1d}$ of Z, we construct a tensor product mesh $\mathcal{T}_h = \mathcal{T}_h^{2d} \times \mathcal{T}_h^{1d}$ of $\Omega = G \times Z$, see Figure 2. The prismatic elements of this mesh can be subdivided if one wishes to work with tetrahedra. We note that the number of elements is $N_e \sim h^{-3}$ independent of $\mu \in (0, 1]$.

Analyzing the elements K of such a mesh, we find that their size is characterized by two independent parameters, namely the mesh size $h_{S,K} \sim h r_K^{1-\mu}$ of the corresponding triangle in $\mathcal{T}_h^{2d}$ and $h_{L,K} \sim h$ of the corresponding interval in $\mathcal{T}_h^{1d}$. Since $h_{S,K} \leq Ch_{L,K}$ for all K we use

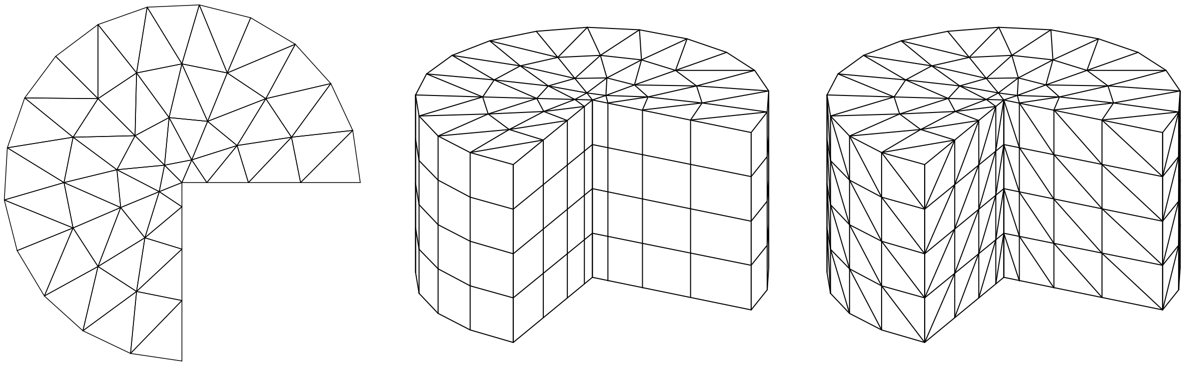

Figure 2: Construction of an anisotropic mesh near the edge.

the indices S (small) and L (large, long). In consequence, the aspect ratio of the element is of the order

$$\frac{h_{L,K}}{h_{S,K}} \sim r_K^{-1+\mu}$$

which becomes *unbounded* for elements close to the edge. Indeed, for such elements there holds $h_{S,K} \sim h\, r_K^{1-\mu} \sim h\, h_{S,K}^{1-\mu}$, i. e. $h_{S,K} \sim h^{1/\mu}$ and $r_K^{-1+\mu} \sim h^{-1/\mu+1}$.

The approximation theory on such so-called anisotropic meshes (also called semi-regular meshes or multi-parameter meshes) is well developed, see [Ape99b]. The estimate

$$\|u - u_h\|_1 \le Ch\|f\|_{0,p} \tag{5}$$

is proved for several cases, e. g. 1) for the Dirichlet problem, $f \in L^p(\Omega)$, $p > 2$ and conforming $\mathcal{P}_1$ elements [AN96], 2) for certain mixed boundary conditions, $f \in L^2(\Omega)$ and conforming $\mathcal{P}_1$ elements [Ape99a], and 3) for Dirichlet boundary conditions, $f \in L^2(\Omega)$ and non-conforming $\mathcal{P}_1$ elements [ANS99]. We underline that the assumption $\lambda > \frac{1}{3}$ is not needed in the case of anisotropic meshes.

The meshes in Figure 2 were constructed by applying a coordinate transformation to a quasi-uniform mesh,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot (x_1^2 + x_2^2)^{(1/\mu-1)/2}. \tag{6}$$

The mesh in Figure 1 was constructed by a *dyadic partition*: Starting with a uniform mesh, all elements that do not satisfy (4), are recursively refined. While the transformation is simpler it has the disadvantage that the spaces V_j corresponding to $h_j = 2^{-j}h_0$, $j = 1, 2, \ldots,$ are not nested. Therefore, standard multi-grid strategies are not applicable in order to solve the resulting system. So we propose to construct the sequence $\mathcal{T}_{h_j}^{2d}$ of two-dimensional graded meshes (as in Figure 2, left) by local transformations in the following way.

Let $\mathcal{T}_{h_0}^{2d}$ be an arbitrary, quasi-uniform, coarse triangular mesh. For $j = 1, 2, \ldots,$ subdivide all elements of $\mathcal{T}_{h_{j-1}}^{2d}$ into four congruent ones (the usual *red refinement*) and apply transformation (6) to the nodes N_i of edges (N_i, P) around the singular vertex P, see Figure 3. All other nodes remain at midpoints of edges of father elements. That means that when a node N_i is inserted on an edge (N_k, P) then the coordinates $(x_1^{(i)}, x_2^{(i)})$ of N_i are locally obtained from the coordinates $(x_1^{(k)}, x_2^{(k)})$ of N_k by

$$\begin{pmatrix} x_1^{(i)} \\ x_2^{(i)} \end{pmatrix} := \left(\frac{1}{2}\right)^{1/\mu} \begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \end{pmatrix}.$$

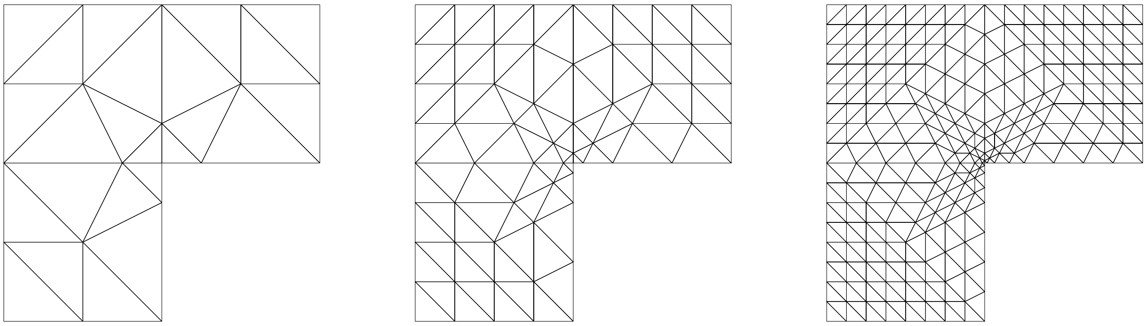

Figure 3: Modified construction of graded meshes.

Note that these local transformations lead also to meshes satisfying relation (4).

The corresponding spaces V_j^{2d} are then nested and a standard multi-grid method is proven to be optimal [Yse86]. For solving the system of the corresponding three-dimensional problem we propose a semi-coarsening strategy, i. e. a mesh hierarchy $\mathcal{T}_{h_j}^{2d} \times \mathcal{T}_h^{1d}$ together with a line smoother [AS00].

Finally, we want to reference [DGP99, Kun99, Sie96] for a-posteriori error estimation on anisotropic meshes.

2 The Stokes problem in an domain with an edge

The solution (u, p) of the Stokes problem

$$-\Delta u + \nabla p = f \quad \text{in } \Omega, \qquad \nabla \cdot u = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{7}$$

has in prismatic domains $\Omega = G \times Z$ a singular behaviour similar to that of the Poisson equation. If the edge is again part of the x_3-axis, and $\lambda \in (\frac{1}{2}, \pi/\omega)$ is the smallest positive solution of

$$\sin(\lambda\omega) + \lambda \sin\omega = 0$$

then u_1, u_2 behave like r^λ, u_3 like $r^{\pi/\omega}$ and p like $r^{\lambda-1}$ near the edge. That means for $\omega > \pi$ that

$$u \notin [H^2(\Omega)]^3, \quad p \notin H^1(\Omega),$$

but, similar to Section 1,

$$\begin{array}{rcll} r^\beta \dfrac{\partial u_i}{\partial x_1},\ r^\beta \dfrac{\partial u_i}{\partial x_2},\ \dfrac{\partial u_i}{\partial x_3} & \in & H^1(\Omega) & \text{for } \beta > 1 - \lambda, \\ r^\beta \dfrac{\partial p}{\partial x_1},\ r^\beta \dfrac{\partial p}{\partial x_2},\ \dfrac{\partial p}{\partial x_3} & \in & L^2(\Omega) & \text{for } \beta > 1 - \lambda, \end{array}$$

[ANS00]. The slightly weaker singularity of u_3 is not separately treated here, since we finally want to approximate all components of the solution on the same mesh. The idea is now to solve problem (7) using a finite element method on anisotropic meshes $\mathcal{T}_h$ as described in Section 1, see also Figure 2.

For the stability of the method it is required that the discrete spaces satisfy an inf-sup condition with a constant independent of the aspect ratio of the elements. Quadrilateral elements have been analyzed by Becker and Rannacher [Bec95, BR95] and by Schötzau, Schwab, and Stenberg [SS98, SSS99]. In particular, the inf-sup condition with a constant independent of the aspect ratio was proved in [Bec95] for stabilized $\mathcal{Q}_1 - \mathcal{P}_0$ and $\mathcal{Q}_1 - \mathcal{Q}_1$ rectangular elements, and in [BR95] for the $\tilde{\mathcal{Q}}_1 - \mathcal{P}_0$ rectangular element. By $\mathcal{Q}_1$ we denote, as usual, the space of bilinear functions, and by $\tilde{\mathcal{Q}}_1$ the non-parametric rotated $\mathcal{Q}_1$ element [RT92]. In [SS98, SSS99] quadrilateral and triangular elements have been considered for the *hp*-version of the finite element method, in particular combinations $\mathcal{Q}_n - \mathcal{Q}_{n-2}$ and $\mathcal{P}_n - \mathcal{P}_{n-2}$, $n \geq 2$. The inf-sup constant does not depend on the aspect ratio, but slightly on n^{-1} ($n^{-1/2}$ for the quadrilaterals and n^{-3} for the triangles). This is compensated by the exponentially good approximation. We remark that all these results were proved for the two-dimensional case.

Well-known triangular elements are the mini element $(\mathcal{P}_1 \oplus \text{bubble}) - \mathcal{P}_1$, the Taylor-Hood element $\mathcal{P}_2 - \mathcal{P}_1$, and its modified form $\mathcal{P}_{1,h/2} - \mathcal{P}_1$. In standard proofs of the inf-sup condition for the isotropic case, the inverse inequality produces a factor h^{-1} which is compensated by a factor h coming from an approximation property. The same proof leads in the anisotropic case to an inf-sup constant depending on the aspect ratio. It has been reported by Russo that the mini element becomes instable on anisotropic meshes [AD99].

A non-conforming method on triangular and tetrahedral meshes is obtained by using the Crouzeix-Raviart ($\mathcal{P}_{-1}$, nonconforming $\mathcal{P}_1$) element for the velocity in combination with piecewise constant pressure. This element was analyzed by Acosta and Duran [AD99] for anisotropic meshes. The inf-sup condition is simple to prove, the challenge is the analysis of the consistency error. Acosta and Duran used the connection to Raviart-Thomas interpolation and succeeded in the case of regular solutions $(u,p) \in [H^2(\Omega)]^3 \times H^1(\Omega)$. Our analysis of this element was performed in [ANS00] independently, by a different approach, and, in particular, for solutions with edge singularities. We will now summarize these results.

Let $X_h := \{v \in L^2(\Omega) : v|_K \in \mathcal{P}_{-1}\ \forall K \in \mathcal{T}_h\}$, $M_h := \{v \in L^2(\Omega) : v|_K \in \mathcal{P}_0\ \forall K \in \mathcal{T}_h\}$, and define bilinear forms

$$a_h(u,v) := \sum_K \sum_{i=1}^{3} \int_K \nabla u_i \cdot \nabla v_i, \qquad b_h(u,q) := -\sum_K \int_K q\, \nabla \cdot u,$$

then we define the finite element solution $(u_h, p_h) \in X_h, M_h$ by

$$\begin{array}{lclcll} a_h(u_h,v_h) & + & b_h(v_h,p_h) & = & (f,v_h) & \forall v_h \in X_h, \\ b_h(u_h,q_h) & & & = & 0 & \forall q_h \in M_h. \end{array}$$

By using the mesh dependent norm $\|\ .\ \|_{1,h}$ defined by $\|v_h\|_{1,h}^2 := a_h(v_h,v_h)$, we are able to prove the inf-sup condition

$$\inf_{q_h \in M_h} \sup_{v_h \in X_h} \frac{b_h(v_h,q_h)}{\|v_h\|_{1,h}\|q_h\|_0} \geq C. \tag{8}$$

It holds for general triangular or tetrahedral meshes.

Denote by $\mathrm{I}_h : [H_0^1(\Omega)]^3 \to X_h$ the Crouzeix-Raviart interpolation operator, defined elementwise by

$$\mathrm{I}_h u|_K \in [\mathcal{P}_1]^3 : \int_F u = \int_F \mathrm{I}_h u \quad \text{for all faces } F \subset \partial K,$$

and by $\mathrm{M}_h : L^2(\Omega) \to M_h$ the local L^2-projector, $\mathrm{M}_h p|_K := (\mathrm{meas}_3 K)^{-1} \int_K p$. Then the following global interpolation error estimates can be proved similarly to the Poisson case:

$$
\begin{aligned}
\|u - \mathrm{I}_h u\|_{1,h} &\leq Ch\,\|f\|_0, && (9)\\
\|p - \mathrm{M}_h p\|_0 &\leq Ch\,\|f\|_0. && (10)
\end{aligned}
$$

Here, we could even use more general meshes than those described in Section 1.

The most difficult part is the estimation of the consistency error. In [ANS00], we finally show that

$$
\sup_{v_h \in V_h} \frac{|a_h(u,v_h) + b_h(v_h,p) - (f,v_h)|}{\|v_h\|_{1,h}} \leq Ch\,\|f\|_0,
$$

where $V_h := \{v_h \in X_h : \nabla \cdot v_h|_K = 0\ \forall K\}$. So we can conclude the approximation error estimate

$$
\|u - u_h\|_{1,h} + \|p - p_h\|_0 \leq Ch\,\|f\|_0.
$$

3 Non-tensor product meshes

In the previous sections, we considered prismatic domains $\Omega = G \times Z$. This simplifies the structure of the solution enormously, since no additional corner singularities appear. In a general polyhedral domain, however, we have to treat both edge and corner singularities. The mesh should be refined towards corners and, anisotropically, towards edges. In [AN98], we formulate a set of assumptions under which the optimal convergence (5) is proved for $p > 2$ in the case of the Poisson problem. Proposals for the construction of meshes that satisfy these assumptions are also given in that paper.

For the Stokes problem with Crouzeix-Raviart elements for the velocity and piecewise constants for the pressure, the inf-sup condition (8) holds for general meshes. The interpolation error estimates (9), (10), can be proved in the same way as for the Poisson problem also for the meshes described in [AN98]. The critical point is the consistency error. The proof in [ANS99, ANS00] heavily relies on the tensor product structure of the mesh. The generalization is an open problem.

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