# Optimal control under reduced regularity

Thomas Apel, Gunter Winkler

*Universität der Bundeswehr München, Fak. für Mathematik und Bauinformatik, 85579 Neubiberg, Germany*

**Abstract**

This paper deals with a linear quadratic optimal control problem with elliptic PDE constraints in three-dimensional domains with singularities. It is proved that the optimal control can be calculated by the finite element method at a rate of $O(h^2)$ provided that the mesh is sufficiently graded. The approximation of this control is computed from a piecewise constant approximation followed by a postprocessing step. Although the results are similar to the two-dimensional case, the proofs changed significantly.

*Key words:* linear quadratic optimal control problem, PDE constraints, finite element method, mesh grading, postprocessing, a-priori error estimates, superconvergence

## 1 Introduction

This paper deals with the numerical solution of the following control-constrained optimal control problem. Let $\Omega \subset \mathbb{R}^3$ be a domain with boundary $\partial\Omega$, $U = L^\infty(\Omega)$ and $[a, b] \subset \mathbb{R}$. Denote by $U_{\mathrm{ad}} = \{u \in U : a \leq u(x) \leq b \text{ a.e. in } \Omega\}$ the set of admissible controls. Let $y_d \in L^\infty(\Omega)$ be the desired state. We consider the optimal control problem

$$J(\bar{u}) = \min_{u \in U_{\mathrm{ad}}} J(u) \tag{1}$$

$$J(u) := \frac{1}{2}\|Su - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)} \tag{2}$$

where the operator $S$ associates the state $y = Su$ to the control $u$ as the weak solution of

$$Ly = u \quad \text{in } \Omega, \qquad y = 0 \quad \text{on } \Gamma = \partial\Omega. \tag{3}$$

The control fulfils pointwise contraints as defined in $U_{\mathrm{ad}}$ and the positive real number $\nu$ is a fixed regularization parameter.

*Email addresses:* thomas.apel@unibw.de (Thomas Apel), gunter.winkler@mathematik.tu-chemnitz.de (Gunter Winkler).

We will investigate the second order elliptic differential operator

$$Ly := \nabla \cdot (A\nabla y) + a \cdot \nabla y + a_0 y \tag{4}$$

with smooth coefficient functions $A(x) \in \mathbb{R}^{3\times 3}$, $a(x) \in \mathbb{R}^3$ and $a_0(x) \in \mathbb{R}$ that satisfy the ellipticity condition

$$\exists m_0 > 0 : m_0|\xi|^2 \leq \xi^T A \xi \qquad \forall x \in \Omega, \forall \xi \in \mathbb{R}^3$$

and the usual condition

$$a_0 - \tfrac{1}{2}\nabla \cdot a \geq 0 \qquad \forall x \in \Omega$$

ensuring coercivity. Additionally we require $A$ to be symmetric.

The numerical solution of the optimal control problem (1)–(3) is currently typically based on a linear or bilinear discretization of the state variable leading to the discrete solution operator $S_h$. For later use we define

$$J_h(u) := \frac{1}{2}\|S_h u - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)}. \tag{5}$$

In the first paper on the discretization, [14], the optimal control $\bar{u}$ is approximated by a piecewise constant function $\bar{u}_h \in U_h^{\mathrm{ad}} := U_h \cap U_{\mathrm{ad}}$, and this discretization is still in use, see, e.g., [10]. The discretized optimal control $\bar{u}_h = \arg\min_{u_h \in U_h^{\mathrm{ad}}} J_h(u_h)$ can be computed from the system

$$\bar{u}_h = \Pi_{[a,b]}\left(-\tfrac{1}{\nu}R_h\bar{p}_h\right), \quad \bar{p}_h = S_h^*(\bar{y}_h - u_d), \quad \bar{y}_h = S_h\bar{u}_h, \tag{6}$$

with the operator $\Pi_{[a,b]}$ projecting into the space $U_{\mathrm{ad}}$ of admissible controls and with an operator $R_h$ that maps continuous functions into the space $U_h$ of piecewise constant functions. The method is said to approximate the optimal control with order $\alpha$ in the discretization parameter $h$ if

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} \leq ch^\alpha,$$

and it is proved in [21] that the convergence order is $\alpha = 1$ if the solution is sufficiently smooth. If the control is approximated by piecewise linear functions this convergence order rises to $\alpha = \frac{3}{2}$, [7,9,11,24–26].

Another approach is, not to discretize the control at all, i. e., the approximate optimal control is $\bar{u}_h = \arg\min_{u \in U_{\mathrm{ad}}} J_h(u)$ and can be computed from the system

$$\bar{u}_h = \Pi_{[a,b]}\left(-\tfrac{1}{\nu}\bar{p}_h\right), \quad \bar{p}_h = S_h^*(\bar{y}_h - u_d), \quad \bar{y}_h = S_h\bar{u}_h. \tag{7}$$

Hinze proved in [17] that the discretization error of the control is bounded by finite element errors,

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} \leq \|(S^* - S_h^*)y_d\|_{L^2(\Omega)} + \|(S^*S - S_h^*S_h)\bar{u}\|_{L^2(\Omega)}, \tag{8}$$

yielding the approximation order $\alpha = 2$ when the approximations of the state and adjoint state are computed by piecewise linear functions and if the state and adjoint state are sufficiently smooth.

Meyer and Rösch were able to show in [23] that the approximation order $\alpha = 2$ can also be achieved with the classical method (6) and piecewise constant approximation $\bar{u}_h$ of the control. They proved the supercloseness result

$$\|R_h\bar{u} - \bar{u}_h\|_{L^2(\Omega)} \leq ch^2 \tag{9}$$

and the error estimate

$$\|\bar{u} - \tilde{u}_h\|_{L^2(\Omega)} \leq ch^2 \tag{10}$$

where the final approximation $\tilde{u}_h$ is constructed by introducing the postprocessing step

$$\tilde{u}_h := \Pi_{[a,b]}\left(-\tfrac{1}{\nu}\bar{p}_h\right) \tag{11}$$

that projects the final adjoint state into the set of admissible controls. The original paper [23] treats the case of convex domains $\Omega \subset \mathbb{R}^2$ under an assumption on the boundary of the active set. Apel, Rösch and Winkler generalized this result for non-convex plane domains in [5]. The article of Rösch and Vexler [27] gives the same result for the Stokes equation in $\Omega \subset \mathbb{R}^3$ under the assumption of full regularity, $\bar{y} \in W^{2,2}(\Omega) \cap W^{1,\infty}(\Omega)$.

In this paper we investigate the three-simensional case in the presence of corner and edge singularities and show for both methods that $\alpha = 2$ can be retained although the state and adjoint state do not have full regularity. The paper does neither target on a comparison of the two approaches nor on new concepts for the numerical solution of optimal control problems.

The key to the proofs is the understanding of the influence of singularities caused by the domain. Regularity results for solutions of elliptic partial differential equations in non-convex domains are given by many authors in the last 40 years. We cite here the monographs by Dauge [13], Grisvard [16], Kufner and Sändig [20], Kozlov, Maz'ya and Roßmann [19]. In Section 2 we recall regularity results for the state equation and some important results of the theory of optimal control.

The reduced regularity of the solution leads to a lower convergence order if standard methods on quasi-uniform meshes are used. The construction of adapted *shape regular* meshes and proofs of finite element approximation results for these meshes were studied in the context of boundary value problems in [2,6]. Section 3 starts with the description of such a discretization. Estimates of the finite element error $Su - S_hu$ are given in the $H^1(\Omega)$- and $L^2(\Omega)$-norms. The former is cited, the latter is derived here since the proof has apparently not been published elsewhere. On the basis of this result, the second order convergence of the approach (7) can then be concluded directly from (8).

We focus in the sequel on the postprocessing approach (6), (11). The proof of the supercloseness and superconvergence results (9) and (10) for this method is more involved and fills Sections 4 and 5 of this paper. It is again based on an assumption on the boundary of the active set, here assumption (37), see page 10. The main parts of the proof are similar to the derivation in [5,23,27] but they have to be adapted to new prerequisites and show a considerable amount of new technical details. The numerical results in Section 6 confirm the expected convergence rates. A final summary is given in Section 7.

## 2 Regularity

In this section we will recall definitions and regularity results from [29] for the solution of the elliptic boundary value problem

$$Ly = f \quad \text{in } \Omega, \qquad y = 0 \quad \text{on } \Gamma = \partial\Omega, \tag{12}$$

in domains with conical points and edges.

We assume that the set $\Omega \subset \mathbb{R}^3$ is a bounded polyhedral domain with 2-dimensional boundary $\partial\Omega$, 1-dimensional non-intersecting edges $M_j \subset \partial\Omega$, and corners $O_i$. The set $M = \bigcup_j \overline{M}_j$ divides $\partial\Omega$ into smooth disjoint connected components, the faces. The set $M$ is called set of singular points or set of singularities. For a more general definition we refer to [13].

The regularity of the solution of partial differential equations on such domains can be expressed with weighted Sobolev spaces. We define

$$V_\beta^{k,p}(\Omega) := \{v \in \mathcal{D}'(\Omega) : \|v\|_{V_\beta^{k,p}(\Omega)} < \infty\},$$

with $k \in \mathbb{N}$, $p \in (1,\infty)$, $\beta \in \mathbb{R}$. By using the standard multi-index notation, the norm is defined by

$$\|v\|_{V_\beta^{k,p}(\Omega)} := \left( \int_\Omega \sum_{|\alpha| \le k} r^{p(\beta-k+|\alpha|)} |D^\alpha v|^p \, dx \right)^{1/p}$$

with $r = r(x) = \operatorname{dist}(M, x)$. We will make use of the fact that

$$c_1 |r^\beta v|_{W^{k,p}(\Omega)} \le \|v\|_{V_\beta^{k,p}(\Omega)} \le c_2 |r^\beta v|_{W^{k,p}(\Omega)}. \tag{13}$$

For proving the desired regularity result, we follow here the outline by Sändig in [29] and we adopt the notation from [4] and [6]. A different approach for proving regularity results in polyhedral domains can be found in [1] and the references therein. The spaces $K_a^\mu(\Omega)$ defined in [1] are the same as is this paper since $\mathcal{K}_a^\mu(\Omega) = V_{\mu-a}^{\mu,2}(\Omega)$, $a \in \mathbb{R}$, $\mu = 0, 1, 2, \ldots$.

**Theorem 2.1** *Let $\Omega \subset \mathbb{R}^3$ be a polyhedron in with corners $O_i$ and edges $M_j$. The weak solution $y$ of* (12) *with right-hand side $f \in L^p(\Omega)$ is contained in $V_\beta^{2,p}(\Omega)$:*

$$\|y\|_{V_\beta^{2,p}(\Omega)} \le c\|f\|_{L^p(\Omega)}$$

*with $\beta > \max\left\{2 - \lambda_{v,i} - \frac{3}{p},\ 2 - \lambda_{e,j} - \frac{2}{p}\right\}$. The values $\lambda_{v,i}$ and $\lambda_{e,j}$ can be computed from a transformed problem near the corner or along the edge, see [18] and [29].*

**Remark 2.2**

*(i) Let $\lambda_v = \min\{\lambda_{v,i}\}$ and $\lambda_e = \min\{\lambda_{e,j}\}$. The above theorem holds for*

$$\beta > \max\left\{2 - \lambda_v - \tfrac{3}{p},\ 2 - \lambda_e - \tfrac{2}{p}\right\} = 2 - 2/p - \min\{\lambda_v + \tfrac{1}{p},\ \lambda_e\}.$$

*For later use we define*

$$\lambda = \min\{\lambda_v + \frac{1}{2},\ \lambda_e\}.$$

*Then Theorem 2.1 holds for $\beta > 1 - \lambda$ if $p = 2$.*

*(ii) There holds $\lambda_v > 0$ and $\lambda_e > \frac{1}{2}$ for many interesting cases, see [6], including the Dirichlet problem.*

*(iii) The embedding $V^{2,p}_\beta(\Omega) \hookrightarrow C(\overline{\Omega})$ is valid if $0 \le \beta < 2 - 3/p$ because this condition allows the embedding $V^{2,p}_\beta(\Omega) \hookrightarrow V^{2-\beta,p}_0(\Omega) \hookrightarrow W^{2-\beta,p}(\Omega) \hookrightarrow C(\overline{\Omega})$, see [28]. Thus the solution $y$ of (12) is continuous if $f \in L^p(\Omega)$ with $p > \frac{1}{\lambda_e}$, in particular*

$$\|y\|_{L^\infty(\Omega)} \le c\|f\|_{L^2(\Omega)} \le c\|f\|_{L^\infty(\Omega)}. \tag{14}$$

*(iv) If $p > 3$ and $y \in V^{2,p}_\beta(\Omega)$ for some $\beta$, we have*

$$\|r^\beta y\|_{W^{1,\infty}(\Omega)} \le c\|r^\beta y\|_{W^{2,p}(\Omega)} \le c\|y\|_{V^{2,p}_\beta(\Omega)}$$

*by the Sobolev embedding theorems and by (13).*

Let us introduce the adjoint problem

$$L^* p = y - y_d \quad \text{in } \Omega, \qquad p = 0 \quad \text{on } \Gamma \tag{15}$$

and denote by $S^*$ the solution operator of this problem, thus $p = S^*(y - y_d)$. Since we can also write

$$p = S^*(Su - y_d) = Pu$$

with an affine operator $P$ we call the solution $p = Pu$ the associated adjoint state to $u$. From now, we will avoid to refer to $p$ as the adjoint state, in order to prevent confusion with the exponent $p$ of the spaces. A consequence of Theorem 2.1 and Remark 2.2(iii) is $Pu \in L^\infty(\Omega) \cap H^1_0(\Omega) \cap V^{2,p}_\beta(\Omega)$, if $p$ and $\beta$ are such that $\beta > \max\left\{2 - \lambda_v - \frac{3}{p},\ 2 - \lambda_e - \frac{2}{p}\right\}$, because $y - y_d \in L^\infty(\Omega)$ holds.

**Corollary 2.3** *If $u, y_d \in L^\infty(\Omega)$ and $\beta > \max\{4/3 - \lambda_e,\ 1 - \lambda_v\}$ then there holds*

$$\|r^\beta \nabla(Pu)\|_{L^\infty(\Omega)} \le \|r^\beta Pu\|_{W^{1,\infty}(\Omega)} \le c\left(\|u\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right). \tag{16}$$

*Further, if $\beta > 1 - \lambda$ the estimate*

$$|Pu|_{V^{2,2}_\beta(\Omega)} \le c\left(\|u\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right) \tag{17}$$

*holds. Finally, $Pu \in W^{1,p}(\Omega)$ if $p < 2/(1 - \lambda_e)$ and $p < 3/(1 - \lambda_v)$.*

**Proof.** From $\beta > \frac{4}{3} - \lambda_e$ we obtain $2/(2 - \beta - \lambda_e) > 2/(2 - \frac{4}{3} + \lambda_e - \lambda_e) = 3$. From $\beta > 1 - \lambda_v$ we conclude similarly $3/(2 - \beta - \lambda_v) > 3/(2 - 1 + \lambda_v - \lambda_v) = 3$. Hence we can choose $p$ with $3 < p < \min\{2/(2 - \beta - \lambda_e),\ 3/(2 - \beta - \lambda_v)\}$ such that all the inequalities $p > 3$, $\beta > 2 - \lambda_e - 2/p$ and $\beta > 2 - \lambda_v - 3/p$ are satisfied. According to Theorem 2.1 we conclude $Pu \in V^{2,p}_\beta(\Omega)$ with the chosen $p$ and $\beta$. Now, we can apply Remark 2.2(iv) which yields

$$\|r^\beta Pu\|_{W^{1,\infty}(\Omega)} \le c\|Pu\|_{V^{2,p}_\beta(\Omega)} \le c\|y - y_d\|_{L^p(\Omega)}$$

We continue the estimate by applying the Sobolev embedding theorem,

$$\|y - y_d\|_{L^p(\Omega)} \le c\|y - y_d\|_{L^\infty(\Omega)} \le c(\|y\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}). \tag{18}$$

The proof is finished by using (14) and the fact that $y$ is a solution of the state equation (3).

For the proof of (17) we do not need the restriction on $p$. Estimate (17) follows from Theorem 2.1 and (18) with $p = 2$ and $\beta > 1 - \lambda$.

From the embedding $V^{2,p}_\beta(\Omega) \hookrightarrow V^{1,p}_{\beta-1}(\Omega)$ we conclude $Pu \in V^{1,p}_{\beta-1}(\Omega)$ if $\beta > 2 - \lambda_e - 2/p$ and $\beta > 2 - \lambda_v - 3/p$. Thus we may choose $\beta = 1$ if $p < 2/(1-\lambda_e)$ and $p < 3/(1-\lambda_v)$ and obtain $Pu \in V^{1,p}_0(\Omega) \hookrightarrow W^{1,p}(\Omega)$. $\square$

In order to formulate the necessary and sufficient first-order optimality condition for the optimal control problem (1), we define the projection

$$\Pi_{[a,b]} f(x) := \max(a, \min(b, f(x))). \tag{19}$$

**Lemma 2.4** *The optimal control problem* (1) *has a unique solution* $\bar u$. *The variational inequality*

$$(\bar p + \nu \bar u,\, u - \bar u)_{L^2(\Omega)} \geq 0 \quad \textit{for all } u \in U_{\mathrm{ad}} \tag{20}$$

*is necessary and sufficient for the optimality of* $\bar u$. *This condition can be expressed equivalently by*

$$\bar u = \Pi_{[a,b]} \left( -\tfrac{1}{\nu} \bar p \right). \tag{21}$$

*Here,* $\bar p = P\bar u$ *denotes the corresponding adjoint state.*

The proof can be found for instance in [21], the key statement is that problem (1) is a convex optimization problem.

**Remark 2.5** *The unique solution* $\bar u$ *of the optimal control problem* (1) *solves the following system of equations*

$$\bar y = S\bar u, \qquad \bar p = S^*(\bar y - y_d), \qquad \bar u = \Pi_{[a,b]} \left( -\tfrac{1}{\nu} \bar p \right). \tag{22}$$

## 3 Discretization and superconvergence results

We consider a family of graded triangulations $(T_h)_{h>0}$ of $\Omega$. All meshes are admissible in Ciarlet's sense [12], in particular shape-regular (isotropic). With $h$ being the global mesh parameter, $\mu \in (0, 1]$ being the grading parameter, and $r_T$ being the distance of a tetrahedron $T$ to $M$,

$$r_T := \inf_{x \in T} \operatorname{dist}(M, x),$$

we assume that the element size $h_T := \operatorname{diam} T$ satisfies

$$\begin{aligned} c_1 h^{1/\mu} &\leq h_T \leq c_2 h^{1/\mu} \quad \text{for } r_T = 0, \\ c_1 h r_T^{1-\mu} &\leq h_T \leq c_2 h r_T^{1-\mu} \text{ for } r_T > 0. \end{aligned} \tag{23}$$

It has been proved in [6] that the number of elements of such a triangulation is of order $h^{-3}$ if $\mu > \frac{1}{3}$. Based on the above triangulation we define spaces of piecewise polynomials

$$\begin{aligned} U_h &= \{ u \in U : u|_T \in \mathcal{P}_0 \ \forall T \in T_h \}, \\ U_h^{\mathrm{ad}} &= U_{\mathrm{ad}} \cap U_h, \\ V_h &= \left\{ v \in C(\overline{\Omega}) : v|_T \in \mathcal{P}_1 \ \forall T \in T_h \text{ and } v_h = 0 \text{ on } \Gamma \right\}. \end{aligned}$$

Now we are able to define the discrete version of the state equation (3). The discretized state $y_h = S_h u$ is the solution of

$$a(y_h, v_h) = (u, v_h)_{L^2(\Omega)} \qquad \forall v_h \in V_h \tag{24}$$

where $a \in H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is the bilinear form

$$a(y, v) = (A\nabla y, \nabla v)_{L^2(\Omega)} + (a \cdot \nabla y + a_0 y, v)_{L^2(\Omega)}.$$

Similarly we define the approximated adjoint state $p_h = S_h^*(y - y_d)$ as the unique solution of

$$a(v_h, p_h) = (y - y_d, v_h) \qquad \forall v_h \in V_h. \tag{25}$$

We further define the affine operator $P_h u = S_h^*(S_h u - y_d)$ that maps a given control $u$ to the approximate adjoint state $p_h = P_h u$.

Finally the discrete optimal control problem is given by

$$\begin{aligned} J_h(\bar{u}_h) &= \min_{u_h \in U_h^{\mathrm{ad}}} J_h(u_h), \\ J_h(u_h) &:= \frac{1}{2}\|S_h u_h - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u_h\|^2_{L^2(\Omega)}. \end{aligned} \tag{26}$$

Similar to equations (22) the optimal control $\bar{u}_h$ is the weak solution of the system (6). The variational inequality

$$(\bar{p}_h + \nu \bar{u}_h, u_h - \bar{u}_h)_{L^2(\Omega)} \geq 0 \quad \text{for all } u_h \in U_h \cap U_{\mathrm{ad}} \tag{27}$$

is necessary and sufficient for the optimality of $\bar{u}_h$, because the discrete problem is still a stricly convex optimization problem.

The next two lemmata collect results from the approximation theory of finite elements which will be used in later theorems.

**Lemma 3.1** *Let $Su$ be the solution of the boundary value problem* (3) *and let $S_h u$ be the solution of* (24)*, then*

$$\|Su - S_h u\|_{H^1(\Omega)} \leq ch^{\alpha}\|u\|_{L^2(\Omega)}, \tag{28}$$
$$\|Su - S_h u\|_{L^2(\Omega)} \leq ch^{2\alpha}\|u\|_{L^2(\Omega)} \tag{29}$$

*holds with $\alpha = \min\left\{\frac{\lambda}{\mu} - \varepsilon,\ 1\right\}$ and $\lambda = \min\{\lambda_e,\ \frac{1}{2} + \lambda_v\}$, $\varepsilon > 0$ arbitrarily small. Additionally, there holds*

$$\|S^* u - S_h^* u\|_{L^2(\Omega)} \leq ch^{2\alpha}\|u\|_{L^2(\Omega)}. \tag{30}$$

**Proof.** The estimate

$$\|Su - S_h u\|_{H^1(\Omega)} \leq c\|Su - I_h Su\|_{H^1(\Omega)} \leq ch^{\alpha}\|u\|_{L^2(\Omega)} \tag{31}$$

was proved in [6]. With the Aubin-Nitsche trick we double the order for the $L^2(\Omega)$-error estimate: Let $w \in V$ be the solution of

$$a(v,w) = (Su - S_h u, v) \quad \forall v \in V$$

and $w_h$ the corresponding finite element solution. By analogy they satisfy

$$\|w - w_h\|_{H^1(\Omega)} \le ch^\alpha \|Su - S_h u\|_{L^2(\Omega)}.$$

Consequently,

$$\begin{aligned}
\|Su - S_h u\|^2_{L^2(\Omega)} &= a(Su - S_h u, w) \\
&= a(Su - S_h u, w - w_h) \\
&\le c\|Su - S_h u\|_{H^1(\Omega)} \|w - w_h\|_{H^1(\Omega)} \\
&\le ch^\alpha \|u\|_{L^2(\Omega)} h^\alpha \|Su - S_h u\|_{L^2(\Omega)}.
\end{aligned}$$

Division by $\|Su - S_h u\|_{L^2(\Omega)}$ yields the assertion of this lemma. The proof of Inequality (30) is similar. $\square$

**Lemma 3.2** *Let $\mu \ge \frac{1}{2}$. The norms of the discrete solution operators $S_h$ and $S_h^*$ are bounded,*

$$\begin{aligned}
\|S_h\|_{L^2(\Omega)\to L^\infty(\Omega)} &\le c, & \|S_h^*\|_{L^2(\Omega)\to L^\infty(\Omega)} &\le c, \\
\|S_h\|_{L^2(\Omega)\to L^2(\Omega)} &\le c, & \|S_h^*\|_{L^2(\Omega)\to L^2(\Omega)} &\le c, \\
\|S_h\|_{L^2(\Omega)\to H^1_0(\Omega)} &\le c, & \|S_h^*\|_{L^2(\Omega)\to H^1_0(\Omega)} &\le c, \\
\|S_h\|_{L^\infty(\Omega)\to L^\infty(\Omega)} &\le c, & \|S_h^*\|_{L^\infty(\Omega)\to L^\infty(\Omega)} &\le c,
\end{aligned}$$

*where $c$ is, as always, independent of $h$.*

**Proof.** We concentrate on the proof of $\|S_h\|_{L^2(\Omega)\to L^\infty(\Omega)} \le c$. The boundedness of $\|S_h\|_{L^2(\Omega)\to L^2(\Omega)}$ and $\|S_h\|_{L^\infty(\Omega)\to L^\infty(\Omega)}$ follows then by the embedding theorem $L^\infty(\Omega) \hookrightarrow L^2(\Omega)$. The boundedness of $\|S_h\|_{L^2(\Omega)\to H^1_0(\Omega)}$ comes from the theory of weak solutions. The respective estimates for $S_h^*$ follow by analogy.

From Remark 2.2(i)–(iii) and Sobolev embedding theorems we conclude that

$$\|Sf\|_{L^\infty(\Omega)} \le c\|Sf\|_{V^{2,2}_\beta(\Omega)} \le c\|f\|_{L^2(\Omega)} \tag{32}$$

with $\frac{1}{2} > \beta > \max\{1 - \lambda_e,\ \frac{1}{2} - \lambda_v\}$. Thus $S$ is a bounded operator from $L^\infty(\Omega)$ into $L^2(\Omega)$. In order to show $\|S_h f\|_{L^\infty(\Omega)} \le c\|f\|_{L^2(\Omega)}$, we choose $T_* \in T_h$ to be the element with the largest norm,

$$\|S_h f\|_{L^\infty(\Omega)} = \|S_h f\|_{L^\infty(T_*)},$$

and continue the estimate with

$$\begin{aligned}
\|S_h f\|_{L^\infty(T_*)} &\le c|T_*|^{-1} \|S_h f\|_{L^1(T_*)} \\
&\le c|T_*|^{-1} \left( \|Sf\|_{L^1(T_*)} + \|(S - S_h)f\|_{L^1(T_*)} \right) \\
&\le c \left( \|Sf\|_{L^\infty(T_*)} + |T_*|^{-1} \|(S - S_h)f\|_{L^1(T_*)} \right)
\end{aligned} \tag{33}$$

It remains to show

$$|T_*|^{-1}\|(S-S_h)f\|_{L^1(T_*)} \le c\|f\|_{L^2(\Omega)}. \tag{34}$$

for isotropic graded meshes with $\mu \ge \frac{1}{2}$, since then we get with (33) and (32) the desired result.

The proof of (34) is carried out by using the Rannacher–Frehse technique, cf. [15]. We define for $e := (S-S_h)f$ the regularized Dirac function

$$\delta^h := \begin{cases} |T_*|^{-1}\mathrm{sgn}(e) & \text{in } T_*, \\ 0 & \text{elsewhere,} \end{cases}$$

and the corresponding regularized Green function $g^h$ as well as its discrete approximation $g_h^h$ as the weak solutions of

$$\begin{aligned} a(v, g^h) &= (\delta^h, v) && \forall v \in V, \\ a(v_h, g_h^h) &= (\delta^h, v_h) && \forall v_h \in V_h, \end{aligned}$$

respectively. According to [8] the three-dimensional Green function satisfies for any fixed $x_+ \in \Omega$.

$$|g(x)| \le c|x - x_+|^{-1},$$

from which we can conclude $\int_{T_*} g(x)\,\mathrm{d}x \le c|T_*|h_{T_*}^{-1}$ after some calculation. This leads to the estimate

$$|g^h(x_+)| = |(\delta^h, g)| \le |T_*^{-1}| \int_{T_*} |g|\,\mathrm{d}x \le ch_{T_*}^{-1},$$

$$\|g^h\|_{L^\infty(\Omega)} \le ch_{T_*}^{-1}.$$

With this result we obtain

$$c\|g^h\|_{H^1(\Omega)}^2 \le a(g^h, g^h) = (\delta^h, g^h) \le \|g^h\|_{L^\infty(\Omega)}\|\delta^h\|_{L^1(\Omega)} \le ch_{T_*}^{-1},$$

$$\|g^h - g_h^h\|_{H^1(\Omega)} \le \|g^h\|_{H^1(\Omega)} \le ch_{T_*}^{-1/2} \le ch^{-1/2\mu}$$

because $h_{T_*} \ge ch^{1/\mu}$. We conclude by using the Galerkin orthogonality and (31) that

$$\begin{aligned} |T_*|^{-1}\|e\|_{L^1(T_*)} &= (\delta^h, e) = a(e, g^h) = a(e, g^h - g_h^h) = a(e - I_h e, g^h - g_h^h) \\ &= a(Sf - I_h Sf, g^h - g_h^h) \\ &\le c\|Sf - I_h Sf\|_{H^1(\Omega)} \cdot \|g^h - g_h^h\|_{H^1(\Omega)} \\ &\le ch^\alpha \|f\|_{L^2(\Omega)} \cdot h^{-\frac{1}{2\mu}} \end{aligned}$$

with $\alpha = \min\{\frac{\lambda}{\mu} - \varepsilon,\, 1\}$. Since $\lambda > \frac{1}{2}$ and $\mu \ge \frac{1}{2}$ we have

$$\alpha - \tfrac{1}{2\mu} = \min\left\{\tfrac{1}{\mu}(\lambda - \tfrac{1}{2}) - \varepsilon,\ 1 - \tfrac{1}{2\mu}\right\} \ge 0$$

and Equation (34) is indeed valid. $\square$

**Corollary 3.3** *Let $u, y_d \in L^2(\Omega)$ be arbitrary functions. The discretization error can be estimated by*

$$\|Pu - P_h u\|_{L^2(\Omega)} \le ch^2 \left( \|u\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)} \right), \tag{35}$$

*provided that the mesh grading parameter satisfies $\mu < \lambda = \min\{\lambda_e, \ \frac{1}{2} + \lambda_v\}$.*

**Proof.** For proving (35), we use

$$Pu - P_h u = S^*(Su - y_d) - S_h^*(S_h u - y_d) = (S^* - S_h^*)(Su - y_d) + S_h^*(S - S_h)u,$$

The assertion (35) follows with the approximation error estimate (29) and (30) in the form

$$\|S^* - S_h^*\|_{L^2(\Omega)\to L^2(\Omega)} \le ch^2, \qquad \|S - S_h\|_{L^2(\Omega)\to L^2(\Omega)} \le ch^2,$$

and the boundedness of $S$ and $S_h^*$ as operators from $L^2(\Omega)$ into $L^2(\Omega)$. □

The construction of the optimal control $\bar{u}$ by system (22) yields that we can assume that the restriction $\bar{u}|_T$ is contained in the space $V_\beta^{2,2}(T)$ for many elements $T$ and all $\beta$ satisfying the requirements of Theorem 2.1. However, we have to prove the following lemmata for all finite elements of the triangulation $T_h$. Therefore we split the domain $\Omega$ in two parts,

$$K_1 := \bigcup_{T \in T_h\colon \bar{u} \notin V_\beta^{2,2}(T)} T, \quad K_2 := \bigcup_{T \in T_h\colon \bar{u} \in V_\beta^{2,2}(T)} T, \quad \beta \text{ from Theorem 2.1.} \tag{36}$$

Clearly, the number of elements in $K_1$ grows for decreasing $h$. Nevertheless, the condition

$$\sum_{T \subset K_1} h_T^2 \le c \tag{37}$$

is fulfilled for isotropic and graded meshes when the boundary of the active set has finite two-dimensional measure. Note that the condition $\sum_{T \subset K_1} h_T^2 \le c$ is sufficient for $|K_1| \le ch$. Another property of such meshes is that the measure of all elements adjacent to the set $M$ of singularities is small. Let

$$K_s = \bigcup_{\{T \in T_h : r_T = 0\}} T \tag{38}$$

and let $n$ be the number of finite elements in $K_s$, that is either a fixed multiple of the number of points if $\dim M = 0$ or of the accumulated length of all edges divided by $h^{1/\mu}$ if $\dim M = 1$, then clearly

$$|K_s| \le cnh^{3/\mu} \le ch^{2/\mu}.$$

For continuous functions $f$ we define the projection $R_h$ into the space $U_h$ by

$$(R_h f)(x) := f(S_T) \quad \text{if } x \in T, \tag{39}$$

where $S_T$ denotes the centroid of the element $T$. In Section 4 we will prove some properties of operator $R_h$ which allow us to formulate the following lemma.

**Lemma 3.4** *Condition* (37) *leads to the estimates*

$$\|S_h\bar u - S_hR_h\bar u\|_{L^2(\Omega)} \le ch^2(\|\bar u\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}), \tag{40}$$
$$\|P_h\bar u - P_hR_h\bar u\|_{L^2(\Omega)} \le ch^2(\|\bar u\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}). \tag{41}$$

The proof is given in Section 4 and is the basis for the following supercloseness result.

**Theorem 3.5** *Let* $\bar u_h$ *be the solution of* (26) *on a family of meshes with grading parameter* $\mu < \min\{\lambda_e, \frac13 + \frac{\lambda_e}{2}, \frac12 + \frac{\lambda_v}{2}\}$. *Assume that* $T_h$ *fulfils the condition* (37). *Then the estimate*

$$\|\bar u_h - R_h\bar u\|_{L^2(\Omega)} \le ch^2\left(\|\bar u\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right) \tag{42}$$

*holds true.*

The proof is given in Section 5. With this preparatory work we can prove the main result for the optimal control problem.

**Theorem 3.6** *Let* $\bar u$, $\bar y$, $\bar p$ *and* $\bar u_h$, $\bar y_h$, $\bar p_h$ *be the solutions of* (22) *and* (6), *respectively, where the family of meshes is graded with parameter* $\mu < \min\{\lambda_e, \frac13 + \frac{\lambda_e}{2}, \frac12 + \frac{\lambda_v}{2}\}$ *and satisfies condition* (37). *Let* $\tilde u_h$ *be the postprocessed control constructed by* (11). *Then the estimates*

$$\|\bar y - \bar y_h\|_{L^2(\Omega)} \le ch^2\left(\|\bar u\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right) \tag{43}$$
$$\|\bar p - \bar p_h\|_{L^2(\Omega)} \le ch^2\left(\|\bar u\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right) \tag{44}$$
$$\|\bar u - \tilde u_h\|_{L^2(\Omega)} \le ch^2\left(\|\bar u\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right) \tag{45}$$

*hold true.*

This conclusion is identically to the one in [5]. For the sake of completeness we sketch it here.

**Proof.** We have

$$\begin{aligned}
\|\bar y - \bar y_h\|_{L^2(\Omega)} &= \|S\bar u - S_h\bar u_h\|_{L^2(\Omega)} \\
&\le \|(S - S_h)\bar u\|_{L^2(\Omega)} + \|S_h(\bar u - R_h\bar u)\|_{L^2(\Omega)} + \|S_h(R_h\bar u - \bar u_h)\|_{L^2(\Omega)}, && (46)\\
\|\bar p - \bar p_h\|_{L^2(\Omega)} &= \|S^*(\bar y - y_d) - S_h^*(\bar y_h - y_d)\|_{L^2(\Omega)} \\
&\le \|(S^* - S_h^*)(\bar y - y_d)\|_{L^2(\Omega)} + \|S_h^*(\bar y - \bar y_h)\|_{L^2(\Omega)}, && (47)\\
\nu\|\bar u - \tilde u_h\|_{L^2(\Omega)} &= \nu\left\|\Pi_{[a,b]}\left(-\tfrac1\nu\bar p\right) - \Pi_{[a,b]}\left(-\tfrac1\nu\bar p_h\right)\right\|_{L^2(\Omega)} \le \|\bar p - \bar p_h\|_{L^2(\Omega)}. && (48)
\end{aligned}$$

The estimate (43) is obtained from (46) by using Lemma 3.1, Lemma 3.4 and Theorem 3.5 combined with Lemma 3.2 and the embedding $L^\infty(\Omega) \hookrightarrow L^2(\Omega)$ where necessary. The estimate (44) can be concluded from (47), Lemma 3.1 and (43). Finally, estimate (45) follows from (48) and (44). $\square$

## 4 Properties of operator $R_h$

This section contains lemmata with properties of the operator $R_h$ defined in (39),

$$(R_h f)(x) := f(S_T) \quad \text{if } x \in T.$$

The point $S_T$ is the centroid of the element $T$.

**Lemma 4.1** *Let $T \in T_h$ and let $R_h$ be the projection defined above. Then there holds*

$$\left| \int_T (f - R_h f)\,\mathrm{d}x \right| \le \begin{cases} c h_T^2 |T|^{1/2} |f|_{W^{2,2}(T)} & \text{for } f \in W^{2,2}(T), \\ c h_T |T|\, |f|_{W^{1,\infty}(T)} & \text{for } f \in W^{1,\infty}(T), \\ c|T|\, \|f\|_{L^\infty(T)} & \text{for } f \in L^\infty(T). \end{cases}$$

**Proof.** The first inequality follows from the fact that the integral vanishes for all linear functions $w \in \mathcal{P}^1(T)$. Thus we can apply the Deny–Lions lemma which gives the desired result after transformation from the reference element.

The proof of the second inequality is similar. Let $\hat{T}$ be the usual three dimensional unit simplex. For any $\hat{w} \in \mathcal{P}^0(\hat{T})$ there holds

$$\int_T (f - R_h f)\,\mathrm{d}x = |T| \int_{\hat{T}} (\hat{f} - \hat{R}\hat{f})\,\mathrm{d}x = |T| \int_{\hat{T}} (\hat{f} - \hat{w}) - \hat{R}(\hat{f} - \hat{w})\,\mathrm{d}x \le c|T| \|\hat{f} - \hat{w}\|_{L^\infty(\hat{T})}$$

Thus we can apply the Deny–Lions lemma which yields

$$\int_T (f - R_h f)\,\mathrm{d}x \le c|T| \inf_{\hat{w} \in \mathcal{P}^0(\hat{T})} \|\hat{f} - \hat{w}\|_{L^\infty(\hat{T})} \le c|T|\, |\hat{f}|_{W^{1,\infty}(\hat{T})} \le c|T| h_T |f|_{W^{1,\infty}(T)}. \tag{49}$$

Finally, we conclude from $\|R_h f\|_{L^\infty(T)} \le \|f\|_{L^\infty(T)}$ that

$$\int_T (f - R_h f)\,\mathrm{d}x \le |T|\, \|f - R_h f\|_{L^\infty(T)} \le 2|T|\, \|f\|_{L^\infty(T)}.$$

This is the third inequality. $\Box$

In order to prove more properties of operator $R_h$ we define the $L^2$-projection operator $\mathrm{Q}_h : L^2(\Omega) \to U_h$ on each element $T$ by

$$\mathrm{Q}_h f\Big|_T = \frac{1}{|T|} \int_T f(x)\,\mathrm{d}x.$$

Since $\mathrm{Q}_h w = w$ for all $w \in \mathcal{P}^0(T)$ we can apply the Deny–Lions lemma which directly implies the inequality

$$\|f - \mathrm{Q}_h f\|_{L^2(T)} \le c h_T |f|_{H^1(T)} \tag{50}$$

for all $f \in H^1(T)$. Further we can conclude by the Cauchy–Schwarz inequality that

$$(f - \mathrm{Q}_h f,\, v)_{L^2(T)} = (f - \mathrm{Q}_h f,\, v - \mathrm{Q}_h v)_{L^2(T)} \le c h_T^2 |f|_{H^1(T)} |v|_{H^1(T)} \tag{51}$$

for any $f, v \in H^1(T)$. One can easily check the identity

$$\mathrm{Q}_h f - R_h f = \frac{1}{|T|} \int_T (f - R_h f)\, \mathrm{d}x \tag{52}$$

using that $R_h f$ is a piecewise constant function.

For the next results we consider functions $f \in W^{1,p}(\Omega)$ with

$$p > 3, \quad p \geq \frac{1}{1-\mu}, \quad p < \frac{2}{1-\lambda_e}, \quad p < \frac{3}{1-\lambda_v}. \tag{53}$$

The first condition is needed to ensure continuity, the second condition is needed in Corollary 4.3, and the last two conditions ensure that the solution of (12) is in $W^{1,p}(\Omega)$, see Corollary 2.3. Note that these inequalities do not conflict if $\mu < \min\{\frac{1}{2} + \frac{\lambda_e}{2}, \frac{2}{3} + \frac{\lambda_v}{3}\}$ which is weaker than condition $\mu < \min\{\lambda_e, \frac{1}{3} + \frac{\lambda_e}{2}, \frac{1}{2} + \frac{\lambda_v}{2}\}$ assumed in Theorems 3.5 and 3.6.

**Lemma 4.2** *The inequality*

$$\|\mathrm{Q}_h f - R_h f\|_{L^2(T)} \leq |T|^{1/2-1/p} h_T |f|_{W^{1,p}(\Omega)}$$

*holds for all $f \in W^{1,p}(T)$ with $p > 3$.*

**Proof.** We have by using (52)

$$\int_T (\mathrm{Q}_h f - R_h f)^2\, \mathrm{d}x = \int_T \left[\frac{1}{|T|}\int_T f - R_h f\, \mathrm{d}\xi\right]^2 \mathrm{d}x = |T|^{-1}\left[\int_T f - R_h f\, \mathrm{d}\xi\right]^2$$

which leads to

$$\|\mathrm{Q}_h f - R_h f\|_{L^2(T)} \leq |T|^{-\frac{1}{2}} \left|\int_T f - R_h f\, \mathrm{d}x\right|.$$

Starting from estimate (49) we conclude by using the embedding $L^\infty(\hat{T}) \hookrightarrow W^{1,p}(\hat{T})$ for $p > 3$ that

$$\begin{aligned}\int_T (f - R_h f)\, \mathrm{d}x &\leq c|T| \inf_{\hat{w} \in \mathcal{P}^0(\hat{T})} \|\hat{f} - \hat{w}\|_{W^{1,p}(\hat{T})} \\ &\leq c|T|\, |\hat{f}|_{W^{1,p}(\hat{T})} \\ &\leq c|T|^{1-1/p} h_T |f|_{W^{1,p}(T)}.\end{aligned}$$

which directly leads to the desired inequality. □

**Corollary 4.3** *Let the mesh be graded with parameter $\mu$. Then*

$$\|\mathrm{Q}_h f - R_h f\|_{L^2(K_s)} \leq ch^2 |f|_{W^{1,p}(K_s)}$$

*holds for all $f \in W^{1,p}(K_s)$ with $p > 3$, $p \geq \frac{1}{1-\mu}$.*

**Proof.** The application of the Hölder inequality and Lemma 4.2 yields

$$\begin{aligned}
\|Q_h f - R_h f\|^2_{L^2(K_s)} &= \sum_{T\in K_s} \|Q_h f - R_h f\|^2_{L^2(T)} \\
&\leq c \sum_{T\in K_s} h_T^{6\left(\frac{1}{2}-\frac{1}{p}\right)+2} |f|^2_{W^{1,p}(T)} \\
&\leq c \left( \sum_{T\in K_s} h_T^{\left(5-\frac{6}{p}\right)\frac{p}{p-2}} \right)^{\frac{p-2}{p}} \cdot \left( \sum_{T\in K_s} |f|^p_{W^{1,p}(T)} \right)^{\frac{2}{p}} \\
&\leq c \left( h_T^{-1} \cdot h_T^{\left(5-\frac{6}{p}\right)\frac{p}{p-2}} \right)^{\frac{p-2}{p}} |f|^2_{W^{1,p}(K_s)}
\end{aligned}$$

By simple computation we see that the exponent of $h_T$ is $4 - 4/p$. Further we get with $1 - 1/p \geq \mu$ as well as $h_T \leq ch^{1/\mu}$ that $h_T^{4-4/p} \leq h_T^{4\mu} \leq ch^4$ and, consequently, the desired result. $\square$

**Corollary 4.4** *Let the mesh be graded with parameter $\mu$. Let $K_s$ from* (38), $K_r = \Omega\backslash\overline{K}_s$, $f \in V^{2,2}_{2-2\mu}(K_r) \cap W^{1,p}(K_s)$. *Then the estimate*

$$\|Q_h f - R_h f\|_{L^2(\Omega)} \leq ch^2 \left( |f|_{V^{2,2}_{2-2\mu}(K_r)} + |f|_{W^{1,p}(K_s)} \right)$$

*holds for all $p$ satisfying* (53).

**Proof.** The estimate on $K_s$ is given by Corollary 4.3. For the estimate on $K_r$ we use the definition of $Q_h$, property (52) and Lemma 4.1,

$$\begin{aligned}
\|Q_h f - R_h f\|^2_{L^2(K_r)} &= \sum_{T\subset K_r} \|Q_h f - R_h f\|^2_{L^2(T)} \\
&= \sum_{T\subset K_r} |T|^{-1} \left| \int_T (f - R_h f)\,\mathrm{d}x \right|^2 \\
&\leq \sum_{T\subset K_r} |T|^{-1} \left[ ch_T^2 |T|^{1/2} |f|_{W^{2,2}(T)} \right]^2 .
\end{aligned}$$

Since $h_T \leq chr_T^{1-\mu}$ and the equivalence of $r_T^{2-2\mu}|f|_{W^{2,2}(T)}$ and $|f|_{V^{2,2}_{2-2\mu}(T)}$ we conclude further

$$\|Q_h f - R_h f\|^2_{L^2(K_r)} \leq ch^4 \sum_{T\subset K_r} |f|^2_{V^{2,2}_{2-2\mu}(T)} \leq ch^4 |f|^2_{V^{2,2}_{2-2\mu}(K_r)}$$

Hence, we have shown the proposition. $\square$

**Lemma 4.5** *Let $\bar{u}$ be the optimal control of problem* (1). *Then the estimate*

$$(Q_h\bar{u} - R_h\bar{u}, v_h)_{L^2(\Omega)} \leq ch^2 \|v_h\|_{L^\infty(\Omega)} \left( \|\bar{u}\|_{L^\infty(\Omega)} + \|\bar{y}_d\|_{L^\infty(\Omega)} \right)$$

*holds for all $v_h \in V_h$ if $\mu < \min\{\lambda_e,\ \frac{1}{3} + \frac{\lambda_e}{2},\ \frac{1}{2} + \frac{\lambda_v}{2}\}$.*

**Proof.** To show the inequality we split the domain in three parts where $\bar{u}$ has different regularity: $K_{1,r} = K_1 \setminus \overline{K}_s$, $K_{2,r} = K_2 \setminus \overline{K}_s$ and $K_s$, see equations (36) and (38). We have

$$\begin{aligned}\int_\Omega v_h(\mathrm{Q}_h\bar{u} - R_h\bar{u})\,\mathrm{d}x &= \sum_{T\in T_h} \int_T v_h \left(\frac{1}{|T|}\int_T (\bar{u} - R_h\bar{u})\,\mathrm{d}\xi\right) \mathrm{d}x \\ &\le \sum_{T\in T_h} \|v_h\|_{L^\infty(T)} \int_T (\bar{u} - R_h\bar{u})\,\mathrm{d}\xi\end{aligned}$$

where we used that $\mathrm{Q}_h\bar{u} - R_h\bar{u}$ is a constant on each $T$. Next we apply Lemma 4.1 on each subdomain to the integral. This yields

$$\begin{aligned}\int_\Omega v_h(\mathrm{Q}_h\bar{u} - R_h\bar{u})\,\mathrm{d}x \le &\sum_{T\subset K_{2,r}} \|v_h\|_{L^\infty(T)} c h_T^2 |T|^{1/2} |\bar{u}|_{W^{2,2}(T)} \\ &+ \sum_{T\subset K_{1,r}} \|v_h\|_{L^\infty(T)} c h_T |T|\, |\bar{u}|_{W^{1,\infty}(T)} \\ &+ \sum_{T\subset K_s} \|v_h\|_{L^\infty(T)} c|T|\, \|\bar{u}\|_{L^\infty(T)}\end{aligned}$$

Using equations (23) and the Cauchy–Schwarz inequality on the first sum as well as the estimates $h_T|T| \le ch_T^4 \le ch_T^2 h^2 r_T^{2-2\mu}$ and $r_T^{2-2\mu}|\bar{u}|_{W^{1,\infty}(T)} \le \|r^{2-2\mu}\nabla\bar{u}\|_{L^\infty(T)}$ on the second sum, we finally get

$$\begin{aligned}\int_\Omega v_h(\mathrm{Q}_h\bar{u} - R_h\bar{u})\,\mathrm{d}x \le \|v_h\|_{L^\infty(\Omega)} \Big( &ch^2 |K_{2,r}|^{1/2}\, |\bar{u}|_{V^{2,2}_{2-2\mu}(K_{2,r})} \\ &+ ch^2 \|r^{2-2\mu}\nabla\bar{u}\|_{L^\infty(K_{1,r})} \sum_{T\subset K_{1,r}} h_T^2 + c|K_s|\, \|\bar{u}\|_{L^\infty(K_s)}\Big)\end{aligned}$$

Next we use that $K_{2,r} \subset \Omega$ is bounded, that the mesh fulfils condition (37) and that $\mu \le 1$ implies $|K_s| \le ch^{2/\mu} \le ch^2$,

$$\begin{aligned}&\int_\Omega v_h(\mathrm{Q}_h\bar{u} - R_h\bar{u})\,\mathrm{d}x \\ &\le ch^2 \|v_h\|_{L^\infty(\Omega)} \left( |\bar{u}|_{V^{2,2}_{2-2\mu}(K_{2,r})} + \|r^{2-2\mu}\nabla\bar{u}\|_{L^\infty(K_{1,r})} + \|\bar{u}\|_{L^\infty(K_s)}\right)\end{aligned}$$

Since $\bar{u}$ is the optimal control of (1) it solves system (22). We can substitute $\bar{u}$ by $-\frac{1}{\nu}\bar{p}$ in the above norms, because $\bar{u}$ is either constant or equal to $-\frac{1}{\nu}\bar{p}$. Finally we extend the domains of the norms and apply Theorem 2.1 and Corollary 2.3 with $\beta = 2 - 2\mu$. Note that $\beta = 2 - 2\mu > \max\{\frac{4}{3} - \lambda_e,\ 1 - \lambda_v\}$ is equivalent to $\mu < \min\{\frac{1}{3} + \frac{\lambda_e}{2},\ \frac{1}{2} + \frac{\lambda_v}{2}\}$. Thus we get

$$\begin{aligned}\int_\Omega v_h(\mathrm{Q}_h\bar{u} - R_h\bar{u})\,\mathrm{d}x &\le ch^2 \|v_h\|_{L^\infty(\Omega)} \left( \frac{1}{\nu}|\bar{p}|_{V^{2,2}_{2-2\mu}(\Omega)} + \frac{1}{\nu}\|r^{2-2\mu}\nabla\bar{p}\|_{L^\infty(\Omega)} + \|\bar{u}\|_{L^\infty(\Omega)}\right) \\ &\le \frac{c}{\nu} h^2 \|v_h\|_{L^\infty(\Omega)} \left( \|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right)\end{aligned}$$

which had to be proven. $\square$

With the help of the $L^2$-projection we are able to prove Lemma 3.4.

**Proof.** [Lemma 3.4] We start with

$$
\begin{aligned}
&\|S_h\bar{u} - S_hR_h\bar{u}\|^2_{L^2(\Omega)} = (S_h\bar{u} - S_hR_h\bar{u}, S_h\bar{u} - S_hR_h\bar{u})_{L^2(\Omega)} \\
&= a(S_h\bar{u} - S_hR_h\bar{u}, P_h\bar{u} - P_hR_h\bar{u}) \\
&= (\bar{u} - R_h\bar{u}, P_h\bar{u} - P_hR_h\bar{u})_{L^2(\Omega)} \\
&= (\bar{u} - \mathrm{Q}_h\bar{u}, P_h\bar{u} - P_hR_h\bar{u})_{L^2(\Omega)} + (\mathrm{Q}_h\bar{u} - R_h\bar{u}, P_h\bar{u} - P_hR_h\bar{u})_{L^2(\Omega)}
\end{aligned} \tag{54}
$$

By definition of $P_h$ the functions $P_h\bar{u}$ and $P_hR_h\bar{u}$ are the solutions of the discretized adjoint equation (25), that means

$$
P_h\bar{u} - P_hR_h\bar{u} = S_h^*(S_h\bar{u} - y_d) - S_h^*(S_hR_h\bar{u} - y_d) = S_h^*(S_h\bar{u} - S_hR_h\bar{u}). \tag{55}
$$

Next we apply estimate (51) to the first term of (54),

$$
\begin{aligned}
\sum_{T\in T_h} (\bar{u} - \mathrm{Q}_h\bar{u}, P_h\bar{u} - P_hR_h\bar{u})_{L^2(T)} &\le c \sum_{T\in T_h} h_T^2 |\bar{u}|_{H^1(T)} |P_h\bar{u} - P_hR_h\bar{u}|_{H^1(T)} \\
&\le ch^2 |\bar{u}|_{H^1(\Omega)} |P_h\bar{u} - P_hR_h\bar{u}|_{H^1(\Omega)}
\end{aligned}
$$

because $h_T^2 \le ch^2$. Since $S_h^*$ is a bounded operator from $L^2(\Omega)$ into $H^1(\Omega)$, see Lemma 3.2, we achieve with (55)

$$
(\bar{u} - \mathrm{Q}_h\bar{u}, P_h\bar{u} - P_hR_h\bar{u})_{L^2(\Omega)} \le ch^2 |\bar{u}|_{H^1(\Omega)} \|S_h\bar{u} - S_hR_h\bar{u}\|_{L^2(\Omega)} \tag{56}
$$

We continue with the second term of (54). We apply Lemma 4.5, Equation (55) and again Lemma 3.2 and get

$$
\begin{aligned}
(\mathrm{Q}_h\bar{u} - R_h\bar{u}, P_h\bar{u} - P_hR_h\bar{u})_{L^2(\Omega)} &\le ch^2 \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right) \|P_h\bar{u} - P_hR_h\bar{u}\|_{L^\infty(\Omega)} \\
&\le ch^2 \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right) \|S_h\bar{u} - S_hR_h\bar{u}\|_{L^2(\Omega)}
\end{aligned}
$$

This estimate gives together with (56) and after division by $\|S_h\bar{u} - S_hR_h\bar{u}\|_{L^2(\Omega)}$ the desired result (40).

Inequality (41) follows from (40) by using (55) and the fact that $S_h^*$ is bounded. $\square$

## 5 Proof of supercloseness of $\bar{u}_h$ and $R_h\bar{u}$

We start by citing a lemma from [5].

**Lemma 5.1** *The inequality*

$$
\nu\|R_h\bar{u} - \bar{u}_h\|^2_{L^2(\Omega)} \le (R_h\bar{p} - \bar{p}_h, \bar{u}_h - R_h\bar{u})_{L^2(\Omega)} \tag{57}
$$

*is valid.*

For the sake of completeness we sketch the proof here.

**Proof.** The optimality condition (20) is true for all $u \in U_{\mathrm{ad}}$. Therefore, we have pointwise a.e.

$$(\bar{p}(x) + \nu\bar{u}(x)) \cdot (u(x) - \bar{u}(x)) \geq 0 \qquad \forall u \in U_{\mathrm{ad}}.$$

Consider any finite element $T$ with center of gravity $S_T$, apply this formula for $x = S_T$ and $u = \bar{u}_h$, integrate over $T$ and accumulate over all $T$. We arrive at

$$(R_h\bar{p} + \nu R_h\bar{u}, \bar{u}_h - R_h\bar{u})_{L^2(\Omega)} \geq 0.$$

Moreover, we can test the optimality condition (27) for $\bar{u}_h$ with the function $R_h\bar{u}$ and get

$$(\bar{p}_h + \nu\bar{u}_h, R_h\bar{u} - \bar{u}_h)_{L^2(\Omega)} \geq 0.$$

We add these two inequalities and obtain an inequality which is equivalent to the formula (57). $\square$

Finally we can use this to prove Theorem 3.5.

**Proof.** [Theorem 3.5] From Lemma 5.1 we get

$$\begin{aligned}
\nu\|R_h\bar{u} - \bar{u}_h\|^2_{L^2(\Omega)} &\leq (R_h\bar{p} - \bar{p}_h,\, \bar{u}_h - R_h\bar{u})_{L^2(\Omega)} \\
&= (R_h\bar{p} - \bar{p},\, \bar{u}_h - R_h\bar{u})_{L^2(\Omega)} + (\bar{p} - P_hR_h\bar{u},\, \bar{u}_h - R_h\bar{u})_{L^2(\Omega)} \\
&\quad + (P_hR_h\bar{u} - \bar{p}_h,\, \bar{u}_h - R_h\bar{u})_{L^2(\Omega)}
\end{aligned} \tag{58}$$

Next we estimate each of the three terms. To the first we apply Corollary 4.4 and Corollary 2.3,

$$\begin{aligned}
(R_h\bar{p} - \bar{p},\, \bar{u}_h - R_h\bar{u})_{L^2(\Omega)} &= (R_h\bar{p} - \mathrm{Q}_h\bar{p},\, \bar{u}_h - R_h\bar{u})_{L^2(\Omega)} + (\mathrm{Q}_h\bar{p} - \bar{p},\, \bar{u}_h - R_h\bar{u})_{L^2(\Omega)} \\
&= (R_h\bar{p} - \mathrm{Q}_h\bar{p},\, \bar{u}_h - R_h\bar{u})_{L^2(\Omega)} \\
&\leq \|R_h\bar{p} - \mathrm{Q}_h\bar{p}\|_{L^2(\Omega)}\|\bar{u}_h - R_h\bar{u}\|_{L^2(\Omega)} \\
&\leq ch^2\left(|\bar{p}|_{V^{2,2}_{2-2\mu}(K_r)} + |\bar{p}|_{W^{1,p}(K_s)}\right)\|\bar{u}_h - R_h\bar{u}\|_{L^2(\Omega)} \\
&\leq ch^2\left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right)\|\bar{u}_h - R_h\bar{u}\|_{L^2(\Omega)}
\end{aligned} \tag{59}$$

The second term can be estimated with the Cauchy–Schwarz inequality

$$(\bar{p} - P_hR_h\bar{u},\, \bar{u}_h - R_h\bar{u})_{L^2(\Omega)} \leq \|P\bar{u} - P_hR_h\bar{u}\|_{L^2(\Omega)}\|\bar{u}_h - R_h\bar{u}\|_{L^2(\Omega)}$$

and by using Corollary 3.3 and Lemma 3.4,

$$\begin{aligned}
\|P\bar{u} - P_hR_h\bar{u}\|_{L^2(\Omega)} &\leq \|P\bar{u} - P_h\bar{u}\|_{L^2(\Omega)} + \|P_h\bar{u} - P_hR_h\bar{u}\|_{L^2(\Omega)} \\
&\leq ch^2\left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right).
\end{aligned} \tag{60}$$

The third term is at most zero because $\bar{p}_h = P_h\bar{u}_h$ and $P_hu = S_h^*(S_hu - y_d)$, and can simply be omitted,

$$
\begin{aligned}
\Big(P_hR_h\bar u - \bar p_h,\, \bar u_h - R_h\bar u\Big)_{L^2(\Omega)} &= \Big(P_hR_h\bar u - P_h\bar u_h,\, \bar u_h - R_h\bar u\Big)_{L^2(\Omega)} \\
&= \Big(S_h(R_h\bar u - \bar u_h),\, S_h(\bar u_h - R_h\bar u)\Big)_{L^2(\Omega)} \le 0. \qquad (61)
\end{aligned}
$$

Thus, (58)–(61) yield

$$\nu\|R_h\bar u - \bar u_h\|^2_{L^2(\Omega)} \le ch^2\left(\|\bar u\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right)\|R_h\bar u - \bar u_h\|_{L^2(\Omega)}$$

which finishes the proof. $\square$

## 6 Numerical Results

Consider the optimal control problem that minimizes the functional (2) where $u \in U_{\mathrm{ad}}$ and where the state $y = Su$ is the weak solution of the boundary value problem

$$-\Delta y + y = u + f \quad \text{in } \Omega, \quad y = 0 \text{ on } \partial\Omega.$$

The domain $\Omega$ can be described in cylindrical coordinates by

$$\Omega = \{(r,\varphi,z) : 0 < r < 1,\, 0 < \varphi < \tfrac{3}{2}\pi,\, 0 < z < 1\}.$$

We choose

$$U_{\mathrm{ad}} = \{u \in U : -0.025 \le u(x) \le 10 \text{ in } \Omega\}.$$

With $\lambda = \lambda_e = 2/3$ and $\alpha = 5/2$ we choose $f$ and $y_d$ such that

$$
\begin{aligned}
\bar y(r,\varphi,z) &= z(1-z)(r^\lambda - r^\alpha)\sin\lambda\varphi, \\
\bar p(r,\varphi,z) &= \nu z(1-z)(r^\lambda - r^\alpha)\sin\lambda\varphi, \\
\bar u(r,\varphi,z) &= \Pi_{[a,b]}\Big(-\tfrac{1}{\nu}\bar p\Big).
\end{aligned}
$$

The result of an example computation is given in Figure 1. The three solid lines show the $L^2$-norms of $\bar u_h - R_h\bar u$, $\bar u - \tilde u_h$ and $\bar u - \bar u_h$, marked with $\square$, $+$ and $\blacksquare$, respectively. The dotted lines show the slope of $O(h) = n^{-1/3}$ and $O(h^2) = n^{-2/3}$. The dashed lines show the $L^\infty$-norm of the three errors and are given only for reference. We see clearly that $\|\bar u - \bar u_h\|_{L^2(\Omega)} = O(h)$ and $\|\bar u_h - R_h\bar u\|_{L^2(\Omega)} = O(h^2)$. The error of $\|\bar u - \tilde u_h\|_{L^2(\Omega)}$ is much smaller than $\|\bar u - \bar u_h\|_{L^2(\Omega)}$ but not on such an ideal line as the other two.

## 7 Conclusions

The important results of this paper include Theorem 3.5,

$$\|\bar u_h - R_h\bar u\|_{L^2(\Omega)} \le ch^2\left(\|\bar u\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right),$$

and Theorem 3.6,

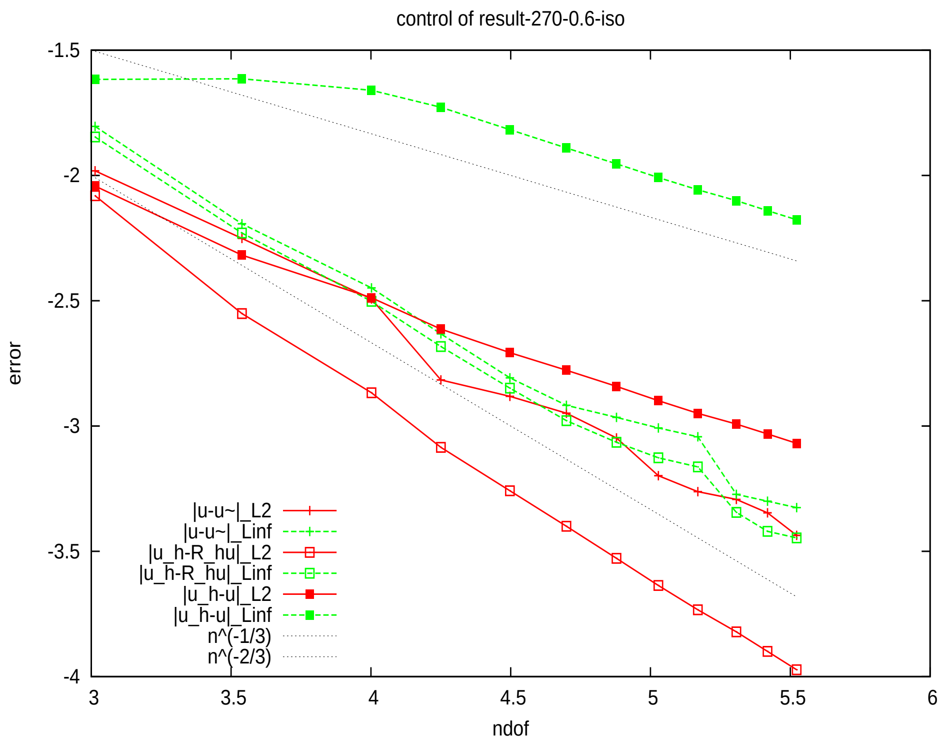


Fig. 1. Errors of the optimal control, $\lambda = \frac{2}{3}$, $\mu = 0.6$, $\alpha = 2.5$, $\nu = 0.01$.

$$
\begin{aligned}
\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} &\le ch^2 \left( \|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)} \right), \\
\|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} &\le ch^2 \left( \|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)} \right), \\
\|\bar{u} - \tilde{u}_h\|_{L^2(\Omega)} &\le ch^2 \left( \|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)} \right)
\end{aligned}
$$

for appropriatly graded meshes. Although the convergence rate is the same as in the two dimensional case presented in [5], the proofs became technically more difficult. In particular, we need the stronger refinement condition $\mu < \frac{1}{3} + \frac{\lambda_e}{2}$ additionally to the condition known from the boundary value problem, $\mu < \lambda_e$. One consequence is that mesh refinement is necessary for all $\lambda_e \le 4/3$. The reason is that only for $\lambda_e > 4/3$ a solution $\bar{y} \in W^{1,\infty}(\Omega)$ is obtained from embedding theorems. For $\lambda_e < 4/3$ we have $y \in W^{2,p}(\Omega)$ with $p < \frac{2}{2-\lambda_e} \le 3$ only, whereas $W^{2,p}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ holds for $p > 3$. That corresponds to [27]. The arguments apply for corner singularities analogously, yielding the condition $\mu < \frac{1}{2} + \frac{\lambda_v}{2}$.

The main challenges in the proofs were first the proof of $\|S_h f\|_{L^\infty(\Omega)} \le c\|f\|_{L^2(\Omega)}$ (Lemma 3.2) where the rough estimates in [5] had to be replaced by a much more careful derivation. Second, the properties of the operator $R_h$ presented in Section 4 needed a completely different approach. Here, the authors especially thank Arnd Rösch and Mariano Mateos for valuable discussions and preliminary copies of their yet unpublished article [22].

The implementation was mostly straightforward. There were only two issues that had to be taken care of. First, the construction of conforming isotropic graded tetrahedral meshes is theoretically simple, but non-trivial to implement. We use the algorithm described in [3]. Second, in order to compute the norm of the error $\|\bar{u} - \tilde{u}_h\|_{L^2(\Omega)}$ one has to numerically integrate non-differentiable functions with high accuracy. The integrator has to identify all tetrahedra where $\bar{u}$ or $\tilde{u}_h$ kink, split them temporarily and approximate the integrals on all parts. However, this procedure is only necessary to compute the error norms presented in Figure 1. It does not belong to the solution strategy investigated here.

## References


[1] B. Ammann, V. Nistor, Weighted Sobolev spaces and regularity for polyhedral domains, ArXiv Mathematics e-prints.
URL `http://eprintweb.org/S/authors/All/ni/Nistor/2`

[2] T. Apel, B. Heinrich, Mesh refinement and windowing near edges for some elliptic problem, SIAM J. Numer. Anal. 31 (1994) 695–708.

[3] T. Apel, F. Milde, Realization and comparison of various mesh refinement strategies near edges, Preprint SPC94_15, TU Chemnitz-Zwickau (1994).

[4] T. Apel, S. Nicaise, The finite element method with anisotropic mesh grading for elliptic problems in domains with corners and edges, Math. Methods Appl. Sci. 21 (1998) 519–549.

[5] T. Apel, A. Rösch, G. Winkler, Optimal control in nonconvex domains, Tech. rep., RICAM Reports (2005-17).
URL `http://www.ricam.oeaw.ac.at/publications/reports/`

[6] T. Apel, A.-M. Sändig, J. R. Whiteman, Graded mesh refinement and error estimates for finite element solutions of elliptic boundary value problems in non-smooth domains., Math. Methods Appl. Sci. 19 (1996) 63–85.

[7] N. Arada, E. Casas, F. Tröltzsch, Error estimates for a semilinear elliptic optimal control problem, Computional Optimization and Approximation 23 (2002) 201–229.

[8] M. Bebendorf, W. Hackbusch, Existence of H-matrix approximants to the inverse FE-matrix of elliptic operators with $L^\infty$-coefficients, Numer. Math. 95 (2003) 1–28.

[9] E. Casas, Using piecewise linear functions in the numerical approximation of semilinear elliptic control problems, submitted.

[10] E. Casas, M. Mateos, F. Tröltzsch, Error estimates for the numerical approximation of boundary semilinear elliptic control problems, Computational Optimization and Applications 31 (2) (2005) 193–219.

[11] E. Casas, F. Tröltzsch, Error estimates for linear-quadratic elliptic control problems, in: V. Barbu (ed.), Analysis and Optimization of Differential Systems, Kluwer Academic Publishers, Boston, 2003.

[12] P. G. Ciarlet, The finite element method for elliptic problems, North-Holland, Amsterdam, 1978, reprinted by SIAM, Philadelphia, 2002.

[13] M. Dauge, Elliptic boundary value problems on corner domains – smoothness and asymptotics of solutions, vol. 1341 of Lecture Notes in Mathematics, Springer, Berlin, 1988.

[14] R. Falk, Approximation of a class of optimal control problems with order of convergence estimates, J. Math. Anal. Appl. 44 (1973) 28–47.

[15] J. Frehse, R. Rannacher, Asymptotic $L^\infty$-error estimates for linear finite element approximations of quasilinear boundary value problems, SIAM J. Numer. Anal. 15 (1978) 418–431.
URL http://link.aip.org/link/?SNA/15/418/1

[16] P. Grisvard, Elliptic problems in nonsmooth domains, vol. 24 of Monographs and Studies in Mathematics, Pitman, Boston, 1985.

[17] M. Hinze, A variational discretization concept in control constrained optimization: The linear-quadratic case, Computational Optimization and Applications 30 (2005) 45–63.

[18] V. A. Kondrat'ev, Boundary value problems for elliptic equations on domains with conical or angular points, Trudy Moskov. Mat. Obshch. 16 (1967) 209–292, in Russian.

[19] V. A. Kozlov, V. G. Maz'ya, J. Roßmann, Elliptic Boundary Value Problems in Domains with Point Singularities, American Mathematical Society, Providence, RI, 1997.

[20] A. Kufner, A.-M. Sändig, Some Applications of Weighted Sobolev Spaces, Teubner, Leipzig, 1987.

[21] K. Malanowski, Convergence of approximations vs. regularity of solutions for convex, control constrained optimal control problems, Appl. Math. Opt. 8 (1981) 69–95.

[22] M. Mateos, A. Rösch, On saturation effects in the Neumann boundary control of elliptic optimal control problems, in preparation.

[23] C. Meyer, A. Rösch, Superconvergence properties of optimal control problems, SIAM J. Control and Optimization 43 (3) (2004) 970–985.

[24] C. Meyer, A. Rösch, $L^\infty$-estimates for approximated optimal control problems, SIAM J. Control and Optimization 44 (5) (2005) 1636–1649.

[25] A. Rösch, Error estimates for parabolic optimal control problems with control constraints, ZAA 23 (2) (2004) 353–376.

[26] A. Rösch, Error estimates for linear-quadratic control problems with control constraints, Optimization Methods and Software 21 (1) (2006) 121–134.

[27] A. Rösch, B. Vexler, Optimal control of the stokes equations: A priori error analysis for finite element discretization with postprocessing, SIAM Journal Numerical Analysis 44 (5) (2006) 1903–1920.

[28] J. Roßmann, Gewichtete Sobolev–Slobodetskiĭ–Räume und Anwendungen auf elliptische Randwertaufgaben in Gebieten mit Kanten, Habilitationsschrift, Universität Rostock (1988).

[29] A.-M. Sändig, Error estimates for finite element solutions of elliptic boundary value problems in non-smooth domains, Z. Anal. Anwend. 9 (2) (1990) 133–153.