Error estimates for control constrained optimal control problems: Discretization with anisotropic finite element meshes

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Abstract A linear-quadratic optimal control problem governed by the Poisson equation with homogenous Dirichlet- or Neumann boundary conditions is investigated. The optimal control has to fulfill box constraints. The domain Ω is assumed to be prismatic with an reentrant edge. The impact of singularities is counteracted by anisotropic mesh grading near the edge. For the piecewise constant approximation of the control followed by a post-processing step a convergence order of two in $L^2(\Omega)$ is shown.

Key Words Elliptic equations, non-convex domains, edge singularities, anisotropic finite elements, error estimates, linear-quadratic optimal control problems, control constraints, superconvergence.

AMS subject classification 65N30, 49M25

1 Introduction

In this paper, we consider the control-constrained optimal control problem

$$J(\bar{u}) = \min_{u \in U^{\mathrm{ad}}} J(u) \tag{1.1}$$

$$J(u) := \frac{1}{2}\|Su - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)} \tag{1.2}$$

where the operator S associates the state $y = Su$ to the control u as the weak solution of

$$Ly = u \quad \text{in } \Omega, \qquad By = 0 \quad \text{on } \Gamma = \partial\Omega. \tag{1.3}$$

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We analyse two different cases, namely pure Dirichlet boundary conditions in the state equation, i.e.

$$L = -\Delta, \quad B = \mathrm{Id}, \tag{1.4}$$

and pure Neumann boundary conditions, i.e.

$$L = -\Delta + \mathrm{Id}, \quad B = \frac{\partial}{\partial n}. \tag{1.5}$$

Robin or mixed boundary conditions are not discussed explicitly here since no further difficulties occur. Here, $\Omega = G \times Z \subset \mathbb{R}^3$ is a domain with boundary $\partial\Omega$, where $G \subset R^2$ is a bounded polygonal domain and $Z := (0, z_0) \subset \mathbb{R}$ is an interval. It is assumed that the cross-section G has only one corner with interior angle $\omega > \pi$ at the origin; thus Ω has only one "singular edge" which is part of the x_3-axis. This is no restriction since the introduced singularity is only of local nature. We set $U = L^\infty(\Omega)$ and denote by $U^{\mathrm{ad}} = \{u \in U : \ u_a \le u(x) \le u_b \text{ a.e. in } \in \Omega\}$ the set of admissible controls. The function $y_d \in C^{0,\sigma}(\bar{\Omega})$, $\sigma \in (0,1)$, is the desired state and the parameter ν a positive real number. Further, we introduce the adjoint problem

$$L^* p = y - y_d \quad \text{in } \Omega, \qquad Bp = 0 \quad \text{on } \Gamma \tag{1.6}$$

and denote by S^* the solution operator of this problem, thus $p = S^*(y - y_d)$. Since one can also write

$$p = S^*(Su - y_d) = Pu$$

with an affine operator P we call the solution $p = Pu$ the associated adjoint state to u. The problem (1.1)-(1.2) admits a unique solution $\bar{u}$ which fulfills the optimality system

$$\begin{aligned} \bar{y} &= S\bar{u}, \\ \bar{p} &= S^*(\bar{y} - y_d), \\ (\nu\bar{u} + \bar{p}, u - \bar{u})_{L^2(\Omega)} &\ge 0 \quad \forall u \in U^{\mathrm{ad}}. \end{aligned} \tag{1.7}$$

The last inequality is equivalent to

$$\bar{u} = \Pi_{[u_a, u_b]}\left(-\frac{1}{\nu}\bar{p}\right) \tag{1.8}$$

where $\Pi_{U^{\mathrm{ad}}}$ is the pointwise projection into the interval $[u_a, u_b]$.

We discretize the optimal control problem based on a finite element approximation of the state variable leading to the discrete solution operator S_h. Results on the discretization of optimal control problems by piecewise constant functions were already given by Falk [12] and Geveci [13]. Malanowski discussed in [17] piecewise constant and piecewise linear discretizations in space for a parabolic problem. In the last years researchers started to investigate numerical schemes for such problems again. Arada, Casas, Tröltzsch, Meyer and Rösch considered piecewise linear approximations of the

control, see [9, 10, 11, 20, 21, 22]. In all that papers the authors proved a convergence order of $k = 1$ or $k = \frac{3}{2}$ in the discretization parameter h,

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} \leq ch^k$$

on quasi-uniform meshes provided the solution is sufficiently smooth. In the variational discretization concept proposed by Hinze [15] the space of admissible controls is not discretized. Instead, the first order optimality condition and the discretization of the state and the adjoint state are utilized to derive an approximate control $\bar{u}_h^v$. It is proved that the discretization error of the control is bounded by finite element errors,

$$\|\bar{u} - \bar{u}_h^v\|_{L^2(\Omega)} \leq \|(S^* - S_h^*)y_d\|_{L^2(\Omega)} + \|(S^*S - S_h^*S_h)\bar{u}\|_{L^2(\Omega)}. \tag{1.9}$$

This gives an approximation order of $k = 2$ for piecewise linear approximations of state and adjoint state as long as the solution is sufficiently smooth. The same result was proved in [8] under reduced regularity assumptions for appropriately graded, isotropic meshes. Another discretization concept was introduced by Meyer and Rösch [19]. The space of admissible controls is discretized by piecewise constant functions, and a post-processing step yields the final approximation,

$$\tilde{u}_h = \Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}\bar{p}_h\right).$$

They proved a convergence order of $k = 2$ for plane, convex domains under the assumption of full regularity in state and adjoint state. Apel, Rösch and Winkler proved in [6] the same result for non-convex plane domains and used local mesh grading. Rösch and Vexler achieved in [23] the same result for the Stokes equation in $\Omega \subset \mathbb{R}^3$ provided that no singularities occur such that $y \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$. The article of Apel and Winkler [8] extends the results for the Poisson equation to general three-dimensional domains, where state and adjoint state may not admit the full regularity. They counteract the impact of singularites, which are caused by reentrant corners and edges, by isotropic, graded meshes and admit a convergence rate of $k = 2$. But already for the state equation itself one can observe, that isotropic mesh refinement along an edge leads to an overrefinement. In order to circumvent this problem anisotropic finite elements were used in [2, 4]. Winkler considered in [25] an anisotropic discretization for the optimal control problem (1.1)–(1.3) with a special type of mixed boundary conditions in the state equation, namely

$$L = \mathrm{Id} \quad \text{in } \Omega, \qquad B = \frac{\partial}{\partial n} \quad \text{on } \Gamma_N, \quad B = \mathrm{Id} \quad \text{on } \Gamma_D.$$

Here, $\Gamma_N = \partial\Omega \cap \{x \in \mathbb{R}^3 : x_3 = 0 \vee x_3 = z_0\}$ and $\Gamma_D = \partial\Omega \backslash \Gamma_N$. The restriction on these boundary conditions was made since the Scott-Zhang type interpolation operator developed in [1] preserves the Dirichlet conditions only on Γ_D and allows an $L^2(\Omega)$-estimate for the finite element error in the state equation only in this situation. In [25] this estimate is a main ingredient of the proof of the convergence order 2 for the $L^2(\Omega)$-error between the approximated control $\tilde{u}_h$ and the optimal control $\bar{u}$. This result was

proven under the mesh grading condition $\mu < \min\{\lambda, \frac{5}{9} + \frac{\lambda}{3}\}$, where μ is the grading parameter and λ the singular exponent. For a detailed definition of these quantities we refer to Sections 2 and 3. This is a stronger condition as actually necessary to get optimal convergence for the state equation itself, where $\mu < \lambda$ is enough.

In this paper here, we extend the results of [25] to the case of pure Dirichlet and pure Neumann boundary conditions as defined in (1.4) and (1.5). The necessary estimates for the finite element error in the state equation are given in a very recent paper by the first two authors, [7]. A challenge in case of Neumann boundary conditions is the fact that the state and the adjoint state do not vanish along the edge. The zero boundary conditions were used in a very explicit manner in the proofs of [25], in particular regularity results were used that are not valid in the Neumann case. Therefore some proofs have to be modified. We further weaken the mesh grading condition given in [25] to $\mu < \lambda$, what is the same as one has to demand to get optimal convergence in the state and adjoint state equation. We have to pay with sligthly more regularity in y_d. So y_d has to be contained in $C^{0,\sigma}(\bar{\Omega})$ and not only in $L^\infty(\Omega)$. As a byproduct we can also weaken the grading condition for isotropic refinement given in [8] (comp. Remark 3.9).

The outline of the paper is as follows. In Section 2 we first recall some regularity results for solutions y of the state equation (1.3) in domains with edges. We further prove that $r^\beta y$ is bounded for $\beta > 1 - \lambda$ as long as the right-hand side is from $C^{0,\sigma}(\bar{\Omega})$. Here, r denotes the distance to the edge. This result is the key to weaken the grading condition from [25]. In Section 3 we discuss the discretization of the optimal control problem and state the main results, namely

$$\|\bar{u} - \bar{u}_h^v\|_{L^2(\Omega)} \leq ch^2 \left(\|\bar{u}\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)}\right) \tag{1.10}$$

and

$$\|\bar{u} - \tilde{u}_h\|_{L^2(\Omega)} \leq ch^2 \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right). \tag{1.11}$$

The estimate (1.10) follows easily by using results of [15] and [7]. Notice, that in the case of variational discretization $y_d \in L^2(\Omega)$ is enough. The details of the proof of estimate (1.11) are given in the Sections 4 and 5. We finish this article by some numerical tests, that illustrate our theoretical findings.

2 Regularity results

First we give regularity results concerning the state equation

$$Ly = f \quad \text{in } \Omega, \qquad By = 0 \quad \text{in } \partial\Omega. \tag{2.1}$$

with L and B from (1.4) and (1.5) respectively. According to [14] the weak solution y of (2.1) can be written for $f \in L^p(\Omega)$, $2 \leq p < \infty$ as a sum of a singular part y_s and a regular part y_r,

$$y = y_s + y_r, \tag{2.2}$$

where $y_r \in W^{2,p}(\Omega)$ and

$$y_s = \xi(r)\gamma(r, x_3) r^\lambda \Theta(\varphi) \quad \text{with } \lambda = \frac{\pi}{\omega}.$$

Here r and φ are polar coordinates in the plane perpendicular to the edge, $\xi(r)$ is a smooth cut-off function and $\Theta(\varphi) = \sin \lambda\varphi$ for the Dirichlet boundary conditions and $\Theta(\varphi) = \cos \lambda\varphi$ for the Neumann boudary conditions. The coefficent function γ can be written as a convolution integral,

$$\gamma(r, x_3) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{r}{r^2 + s^2} q(x_3 - s)\, \mathrm{ds}$$

where the smoothness of q can be characterized in Besov spaces depending on λ.

Lemma 2.1. *Let y be the weak solution of (2.1) for a right-hand side $f \in L^p(\Omega)$, $2 \le p < \infty$. For the singular part y_s the inequalities*

$$\|r^\beta \partial_{ij} y_s\|_{L^p(\Omega)} + \|\partial_{3i} y_s\|_{L^p(\Omega)} + \|\partial_{33} y_s\|_{L^p(\Omega)} \le c\|f\|_{L^p(\Omega)}, \quad i, j = 1, 2 \tag{2.3}$$

$$\|r^{\beta-1} \partial_i y_s\|_{L^p(\Omega)} + \|r^{-1} \partial_3 y_s\|_{L^p(\Omega)} \le c\|f\|_{L^p(\Omega)}, \quad i = 1, 2 \tag{2.4}$$

$$\|r^{\beta-2} y_s\|_{L^p(\Omega)} \le c\|f\|_{L^p(\Omega)} \tag{2.5}$$

are valid for

$$\beta > 2 - \frac{2}{p} - \lambda \quad \text{if } \ 1 - \frac{2}{p} < \lambda \le 2 - \frac{2}{p} \qquad \text{and}$$

$$\beta = 0 \quad \text{if } \ \lambda > 2 - \frac{2}{p}.$$

For the regular part y_r the estimate

$$\|y_r\|_{W^{2,p}(\Omega)} \le c\|f\|_{L^p(\Omega)} \tag{2.6}$$

holds.

Proof. In [3, Section 2.1] the assertions (2.3)–(2.5) are proved for the Dirichlet problem. In order to get the estimates for the Neumann problem one just has to replace $\sin\left(\frac{j\pi\varphi}{\omega}\right)$ by $\cos\left(\frac{j\pi\varphi}{\omega}\right)$ in that proof. Expression (2.6) follows from [14, Theorem 6.6]. □

Remark 2.2. *For the Dirichlet problem the inequalities (2.3)–(2.5) are also valid for the regular part y_r (see [16]). This is not the case for the Neumann problem since the regular part needs not to vanish at the edge.*

It is well known that the weak solution y of the boundary value problem (2.1) is not contained in the space $W^{1,\infty}(\Omega)$. Instead, one has $r^\beta \nabla y \in L^\infty(\Omega)$ with a suitable weight β. A reasonable attempt to determine an appropriate value for the weight β is the use of Sobolev embedding theorems and Lemma 2.1. This yields the condition $\beta > \frac{4}{3} - \lambda$. For details on this we refer to [25]. But since $y \sim r^\lambda$ and consequently $\nabla y \sim r^{\lambda-1}$ one can expect, that a weight $\beta > 1 - \lambda$ is large enough. In the following lemma we show, that this is actually true.

Lemma 2.3. *Let y be the weak solution of (2.1) with a right-hand side $f \in C^{0,\sigma}(\bar{\Omega})$, $\sigma \in (0,1)$. Then the estimates*

$$\|r^\beta y\|_{L^\infty(\Omega)} \le c\|f\|_{C^{0,\sigma}(\bar{\Omega})}, \quad \beta > 1-\lambda \tag{2.7}$$

$$\|\partial_3 y\|_{L^\infty(\Omega)} \le c\|f\|_{C^{0,\sigma}(\bar{\Omega})} \tag{2.8}$$

hold true.

Proof. In order to prove the assertion (2.7), we use the results from [18, Subsection 5.3]. From Theorem 5.1 and its proof in that paper, one has the a priori estimate

$$\|y\|_{C^{2,\sigma}_{\gamma,\delta}(\Omega)} \le c\|f\|_{C^{0,\sigma}_{\gamma,\delta}(\Omega)}. \tag{2.9}$$

In the case of our prismatic domain the norm in $C^{l,\sigma}_{\gamma,\delta}(\Omega)$ that is given in [18] reduces to

$$\begin{aligned}\|y\|_{C^{l,\sigma}_{\gamma,\delta}(\Omega)} = &\sum_{|\alpha|\le l} \sup_{x\in\Omega} (\rho_1(x)\rho_2(x))^{\gamma-l-\sigma+|\alpha|} \left(\frac{r(x)}{\rho(x)}\right)^{H(\delta-l-\sigma+|\alpha|)} |\partial^\alpha y(x)| \\ &+ \sum_{k=1}^{2} \sum_{|\alpha|=l-k_1} \sup_{x_1,x_2\in\Omega} \rho_k(x_1)^{\gamma-\delta} \frac{|\partial^\alpha y(x_1) - \partial^\alpha y(x_2)|}{|x-y|^{k_1+\sigma-\delta}} \\ &+ \sum_{|\alpha|=l} \sup_{|x_1-x_2|<r(x_1)/2} \rho_1(x_1)^\gamma \rho_2(x_1)^\gamma \left(\frac{r(x_1)}{\rho(x_1)}\right)^\delta \frac{|\partial^\alpha y(x_1) - \partial^\alpha y(x_1)|}{|x_1-x_2|^\sigma}.\end{aligned} \tag{2.10}$$

The second term only appears in case of Neumann boundary conditions. Here, $\rho_1(x)$ and $\rho_2(x)$ denote the distance of x to the corners, $r(x)$ is the distance of x to the edge and $\rho(x) = \min(\rho_1(x), \rho_2(x))$. Further, $k_1 = [\delta - \sigma] + 1$, where $[\delta - \sigma]$ denotes the greatest integer less or equal to $\delta - \sigma$. The function H is defined as $H(t) = t$ for Dirichlet boundary conditions and as $H(t) = \max(t, 0)$ for Neumann boundary conditions. For the prismatic domain we can choose $\gamma = \delta$ with the conditions

$$2 - \lambda + \sigma < \gamma < 2 + \sigma \tag{2.11}$$

and $\gamma - \sigma \neq 1$. Now we can reduce our considerations concerning the norm in $C^{2,\sigma}_{\gamma,\delta}(\Omega)$ on the first term and $|\alpha| = 1$. Taking $\gamma = \delta$ into account, the relevant part is

$$M := \sum_{|\alpha|=1} \sup_{x\in\Omega} (\rho_1(x)\rho_2(x))^{\gamma-1-\sigma} \left(\frac{r(x)}{\rho(x)}\right)^{H(\gamma-1-\sigma)} |\partial^\alpha y(x)|.$$

Using inequality (2.11) it follows

$$\gamma - 1 - \sigma > 2 - \lambda - 1 = 1 - \lambda > 0 \tag{2.12}$$

since $\lambda \in \left(\frac{1}{2}, 1\right)$. Therefore $H(\gamma-1-\sigma) = \gamma-1-\sigma$ in both cases, Dirichlet and Neumann boundary condition. Now we introduce the domains $\Omega_1 = \{x \in \Omega, \rho(x) = \rho_1(x)\}$ and $\Omega_2 = \{x \in \Omega, \rho(x) = \rho_2(x)\}$. For every α with $|\alpha| = 1$, one can write

$$\begin{aligned}
&\sup_{x\in\Omega} (\rho_1(x)\rho_2(x))^{\gamma-1-\sigma} \left(\frac{r(x)}{\rho(x)}\right)^{\gamma-1-\sigma} |\partial^\alpha y(x)| \\
&\qquad \geq \sup_{x\in\Omega_1} (\rho_1(x)\rho_2(x))^{\gamma-1-\sigma} \left(\frac{r(x)}{\rho(x)}\right)^{\gamma-1-\sigma} |\partial^\alpha y(x)| \\
&\qquad = \sup_{x\in\Omega_1} \rho_2(x)^{\gamma-\sigma-1} r(x)^{\gamma-\sigma-1} |\partial^\alpha y(x)| \\
&\qquad \geq c \cdot \sup_{x\in\Omega_1} r(x)^{\gamma-\sigma-1} |\partial^\alpha y(x)| \qquad (2.13)
\end{aligned}$$

since $\rho_2(x) \geq \frac{1}{2}$ for $x \in \Omega_1$. Analogously one has

$$\sup_{x\in\Omega} (\rho_1(x)\rho_2(x))^{\gamma-1-\sigma} \left(\frac{r(x)}{\rho(x)}\right)^{\gamma-1-\sigma} |\partial^\alpha y(x)| \geq c \cdot \sup_{x\in\Omega_2} r(x)^{\gamma-\sigma-1} |\partial^\alpha y(x)|. \qquad (2.14)$$

The estimates (2.13) and (2.14) yield

$$M \geq \|r^{\gamma-\sigma-1} \nabla y\|_{L^\infty(\Omega)}.$$

This entails for $\beta := \gamma - \sigma - 1$

$$\|r^\beta \nabla y\|_{L^\infty(\Omega)} \leq c\|y\|_{C^{2,\sigma}_{\gamma,\gamma}(\Omega)} \leq c\|f\|_{C^{0,\sigma}_{\gamma,\gamma}(\Omega)}, \quad \beta > 1 - \lambda, \qquad (2.15)$$

where we have used (2.9) and (2.12). In the following lines, we show

$$C^{0,\sigma}(\bar{\Omega}) \hookrightarrow C^{0,\sigma}_{\gamma,\gamma}(\Omega) \text{ for } \gamma - \sigma \geq 0. \qquad (2.16)$$

The first term in the norm definition (2.10) yields for $l = 0$

$$\sup_{x\in\Omega} \rho_1(x)^{\gamma-\sigma} \rho_2(x)^{\gamma-\sigma} \left(\frac{r(x)}{\rho(x)}\right)^{H(\gamma-\sigma)} |y(x)| \leq c \cdot \sup_{x\in\Omega} r(x)^{\gamma-\sigma} |y(x)|$$

with the same argumentation as above. Analogously, the third term results in

$$\sup_{|x_1-x_2|<r(x_1)/2} \rho_1^\gamma \rho_2^\gamma \left(\frac{r(x)}{\rho(x)}\right)^\gamma \frac{|y(x_1)-y(x_2)|}{|x_1-x_2|^\sigma} \leq c \cdot \sup_{|x_1-x_2|<r(x_1)/2} r(x_1)^\gamma \frac{|y(x_1)-y(x_2)|}{|x_1-x_2|^\sigma}.$$

With $\gamma > \gamma - \sigma > 0$ these two estimates yield (2.16). Therefore the assertion (2.7) follows from (2.15). According to Lemma 2.1, one has $\partial_3 y \in W^{1,p}(\Omega)$. For $p > 3$ the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ is valid. Therefore we can conclude

$$\|\partial_3 y\|_{L^\infty(\Omega)} \leq c\|\partial_3 y\|_{W^{1,p}(\Omega)} \leq c\|f\|_{L^p(\Omega)} \leq c\|f\|_{C^{0,\sigma}(\bar{\Omega})}$$

what is exactly assertion (2.8). □

The last lemma yields directly a regularity result for the adjoint state p.

Corollary 2.4. *Consider the optimality system* (1.7) *with a desired state* $y_d \in C^{0,\sigma}(\bar{\Omega})$, $\sigma \in (0,1)$. *If* $\beta > 1-\lambda$ *then there holds for* $i = 1, 2$

$$\|r^\beta \partial_i \bar{p}\|_{L^\infty(\Omega)} \leq c \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})} \right), \tag{2.17}$$

$$\|\partial_3 \bar{p}\|_{L^\infty(\Omega)} \leq c \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})} \right). \tag{2.18}$$

Proof. From inequality (2.7) one has for $\sigma \in (0,1)$ the estimate

$$\|r^\beta \partial_i \bar{p}\|_{L^\infty(\Omega)} \leq c\|\bar{y} - y_d\|_{C^{0,\sigma}} \leq c \left(\|\bar{y}\|_{C^{0,\sigma}(\bar{\Omega})} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})} \right) \tag{2.19}$$

where we have used the triangle inequality in the last step. For the proof of assertion (2.17) it remains to show that the estimate

$$\|\bar{y}\|_{C^{0,\sigma}(\bar{\Omega})} \leq c\|\bar{u}\|_{L^\infty(\Omega)} \tag{2.20}$$

is valid for some $\sigma \in (0,1)$. In the following we assume $\sigma < \lambda$. For $0 < \gamma < 2 - \frac{3}{p} - \sigma$ with p specified below the inclusion

$$V_\gamma^{2,p}(\Omega) \hookrightarrow V_0^{2-\gamma,p}(\Omega) \hookrightarrow W^{2-\gamma,p}(\Omega) \hookrightarrow C^{0,\sigma}(\bar{\Omega}) \tag{2.21}$$

is valid. For the first embedding we have used [24, Lemma 1.2]. The other inclusions follow by the Sobolev embedding theorems and the fact that $2 - \gamma - \frac{3}{p} > \sigma$. Taking the decomposition $\bar{y} = \bar{y}_r + \bar{y}_s$ into account one can conclude from Lemma 2.1 $\bar{y}_r \in W^{2,p}(\Omega)$ and $\bar{y}_s \in V_\gamma^{2,p}(\Omega)$ for $\gamma > 2 - \frac{2}{p} - \lambda$. In order to be able to find γ such that

$$2 - \frac{2}{p} - \lambda < \gamma < 2 - \frac{3}{p} - \sigma, \tag{2.22}$$

we have to choose p such that $\frac{1}{p} < \lambda - \sigma$. Since $\sigma < \lambda$, the condition $p > \frac{1}{\lambda - \sigma}$ guarantees the existence of a weight λ satisfying (2.22). With such a weight γ we can write for $p > \max\left(\frac{1}{\lambda-\sigma}, \frac{3}{2-\sigma}\right)$ and $\sigma < \lambda$

$$\begin{aligned} \|\bar{y}\|_{C^{0,\sigma}(\bar{\Omega})} &\leq c \left(\|\bar{y}_s\|_{C^{0,\sigma}(\bar{\Omega})} + \|\bar{y}_r\|_{C^{0,\sigma}(\bar{\Omega})} \right) \\ &\leq c \left(\|\bar{y}_s\|_{V_\gamma^{2,p}(\Omega)} + \|\bar{y}_r\|_{W^{2,p}(\Omega)} \right) \\ &\leq c\|\bar{u}\|_{L^p(\Omega)} \leq c\|\bar{u}\|_{L^\infty(\Omega)}, \end{aligned}$$

where we have used the embeddings (2.21) and $W^{2,p}(\Omega) \hookrightarrow C^{0,\sigma}(\bar{\Omega})$ for $p > \frac{3}{2-\sigma}$ as well as Lemma 2.1. This proves inequality (2.20). The assertion (2.17) follows then from estimate (2.19). If we use estimate (2.8) and inequality (2.19), one can conclude

$$\|\partial_3 \bar{p}\|_{L^\infty(\Omega)} \leq c\|\bar{y} - y_d\|_{C^{0,\sigma}(\bar{\Omega})} \leq c \left(\|\bar{y}\|_{C^{0,\sigma}(\bar{\Omega})} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})} \right).$$

where we have used the triangle inequality in the last step. Inequality (2.20) yields then the assertion (2.18). □

3 Discretization

In this section we introduce a discretization concept for the optimal control problem (1.1)–(1.2). Based on a supercloseness result we prove the main result of this paper, namely the superconvergence of the postprocessed approximated control to the optimal solution. This is stated in Theorem 3.8.

To this end we define a family of meshes $\mathcal{T}_h = \{T\}$ of tensor product type (comp. [1], [7]). First, we introduce a graded, isotropic triangulation $\{\tau\}$ in the two-dimensional domain G. The elements are triangles. With h being the global mesh parameter, $\mu \in (0, 1]$ being the grading parameter and r_τ being the distance to the corner,

$$r_\tau := \inf_{(x_1,x_2)\in\tau} (x_1^2 + x_2^2)^{1/2},$$

the element size $h_\tau = \operatorname{diam} \tau$ is assumed to satisfy

$$h_\tau \sim \begin{cases} h^{1/\mu} & \text{for } r_\tau = 0, \\ h r_\tau^{1-\mu} & \text{for } 0 < r_\tau \le R, \\ h & \text{for } r_\tau > R. \end{cases}$$

Here, R is some constant. From this graded two-dimensional mesh we build a three-dimensional mesh of pentahedra by extruding the triangles τ in x_3-direction with uniform mesh size h. In order to generate an anisotropic graded tetrahedral mesh, we divide each of these pentrahedra into tetrahedra. We can characterize the elements T of such a mesh by the three mesh sizes $h_{T,1}$, $h_{T,2}$ and $h_{T,3}$, where $h_{T,i}$ is the length of the projection of T on the x_i-axis, $i = 1, 2, 3$. In detail, with r_T being the distance of the element T to the edge,

$$r_T := \inf_{x\in T} (x_1^2 + x_2^2)^{1/2},$$

the element sizes satisfy

$$\begin{aligned} &h_{T,i} \sim h^{1/\mu} && \text{for } r_T = 0, \\ &h_{T,i} \sim h r_T^{1-\mu} && \text{for } r_T > 0, \\ &h_{T,3} \sim h, \end{aligned} \tag{3.1}$$

for $i = 1, 2$.

In the following we will frequently use the multi-index notation. For $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3$ we denote

$$\begin{aligned} |\alpha| &= \alpha_1 + \alpha_2 + \alpha_3, \\ \partial^\alpha f &= \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3} f, \\ h_T^\alpha &= h_{T,1}^{\alpha_1} h_{T,2}^{\alpha_2} h_{T,3}^{\alpha_3}. \end{aligned}$$

Based on the above triangulation we define spaces of piecewise polynomials

$$\begin{aligned} U_h &= \{u \in U : u|_T \in \mathcal{P}_0 \ \forall T \in T_h\}, \\ U_h^{\mathrm{ad}} &= U^{\mathrm{ad}} \cap U_h, \\ V_h &= \left\{v \in C(\bar\Omega) : v|_T \in \mathcal{P}_1 \ \forall T \in T_h \text{ and } v_h = 0 \text{ on } \Gamma_D\right\}. \end{aligned}$$

Now we are able to formulate a discrete version of the state equation (1.3). The approximated state $y_h = S_h u$ is the unique solution of

$$a(y_h, v_h) = (u, v_h)_{L^2(\Omega)} \qquad \forall v_h \in V_h$$

where $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is the bilinear form

$$a(y, v) = (\nabla y, \nabla v)_{L^2(\Omega)} + k \cdot (y, v)_{L^2(\Omega)}.$$

One sets $k = 0$ in case of Dirichlet boundary conditions (1.4) and $k = 1$ in case of Neumann boundary conditions (1.5). Similiary we define the approximated adjoint state $p_h = S_h^*(y - y_d)$ as the unique solution of

$$a(v_h, p_h) = (y - y_d, v_h)_{L^2(\Omega)} \qquad \forall v_h \in V_h.$$

We further denote by $P_h u = S_h^*(S_h u - y_d)$ the affine operator that maps a given control u to the corresponding approximate adjoint state p_h.

Finally, the discretized optimal control problem reads as

$$J_h(\bar{u}_h) = \min_{u_h \in U_h^{\mathrm{ad}}} J_h(u_h) \tag{3.2}$$

$$J_h(u_h) := \frac{1}{2}\|S_h u_h - y_d\|_{L^2(\Omega)} + \frac{\nu}{2}\|u_h\|_{L^2(\Omega)}. \tag{3.3}$$

This strictly convex optimization problem admits a unique solution $\bar{u}_h$, that satisfies the first order optimality conditions

$$\begin{aligned} \bar{y}_h &= S_h \bar{u}_h, \\ \bar{p}_h &= S_h^* \left(\bar{y}_h - y_d\right), \\ (\nu \bar{u}_h + \bar{p}_h, u_h - \bar{u}_h)_{L^2(\Omega)} &\geq 0 \qquad \forall u_h \in U_h^{\mathrm{ad}}. \end{aligned} \tag{3.4}$$

As in the continuous case these conditions are necessary and sufficient.

Let us now collect some results from the finite element theory.

Lemma 3.1. *Let $u \in L^2(\Omega)$ be an arbitrary function and the mesh be graded according to* (3.1) *with parameter $\mu < \lambda$. Then the estimate*

$$\|Su - S_h u\|_{L^2(\Omega)} \leq ch^2 \|u\|_{L^2(\Omega)}$$

is valid.

Proof. This lemma is proved in [7]. □

Due to fact that we do not operate on quasi-uniform meshes the boundedness of the operator S_h is not obvious. The following lemma is proved in [25, Subsection 3.6] by using Green function techniques.

Lemma 3.2. *Let T_h be an anisotropic, graded mesh of a prismatic domain with parameter $\mu < \lambda$. The norms of the discrete solution operators S_h and S_h^* are bounded,*

$$\begin{aligned} \|S_h\|_{L^2(\Omega)\to L^\infty(\Omega)} &\le c, & \|S_h^*\|_{L^2(\Omega)\to L^\infty(\Omega)} &\le c,\\ \|S_h\|_{L^2(\Omega)\to L^2(\Omega)} &\le c, & \|S_h^*\|_{L^2(\Omega)\to L^2(\Omega)} &\le c,\\ \|S_h\|_{L^2(\Omega)\to H_0^1(\Omega)} &\le c, & \|S_h^*\|_{L^2(\Omega)\to H_0^1(\Omega)} &\le c,\\ \|S_h\|_{L^\infty(\Omega)\to L^\infty(\Omega)} &\le c, & \|S_h^*\|_{L^\infty(\Omega)\to L^\infty(\Omega)} &\le c, \end{aligned}$$

where c is independent of h.

We can use this result to prove an L^2-error estimate for the finite element approximation of the adjoint state.

Lemma 3.3. *Let u, $y_d \in L^2(\Omega)$ be arbitrary functions. Then the inequality*

$$\|Pu - P_h u\|_{L^2(\Omega)} \le ch^2 \left(\|u\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)}\right)$$

holds true.

Proof. One can write

$$Pu - P_h u = S^*(Su - y_d) - S_h^*(S_h u - y_d) = (S^* - S_h^*)(Su - y_d) + S_h^*(S - S_h)u.$$

Then the assertion follows directly from Lemma 3.1 and the boundedness of S and S_h^* as operators from $L^2(\Omega)$ to $L^2(\Omega)$ (comp. Lemma 2.1 and Lemma 3.2). □

Remark 3.4. *From Lemma 3.1 and Lemma 3.3 it follows directly*

$$\begin{aligned} \|(S^*S - S_h^*S_h)\bar u\|_{L^2(\Omega)} &= \|(P - P_h)\bar u + (S^* - S_h^*)y_d\|_{L^2(\Omega)}\\ &\le ch^2 \left(\|\bar u\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)}\right) \end{aligned}$$

for $\mu < \lambda$. This yields second order convergence for the method proposed by Hinze (comp. (1.9) *and [15, Theorem 2.4]).*

From the projection formula (1.8) one can see, that there may be elements where the optimal control $\bar u$ admits kinks. For such an element T one cannot assume that the restriction $\bar u|_T$ is contained in $V_\beta^{2,2}(T)$. Consequently, a special treatment is necessary during the error analysis. Therefore we split the domain Ω in two parts,

$$K_1 := \bigcup_{T\in T_h : \bar u \notin V_\beta^{2,2}(T)} T, \quad K_2 := \bigcup_{T\in T_h : \bar u \in V_\beta^{2,2}(T)} T.$$

Clearly, the number of elements in K_1 grows for decreasing h. Nevertheless, it is quite reasonable to assume that the boundary of the active set has finite two-dimensional measure, i.e.

$$|K_1| \le ch. \tag{3.5}$$

Notice, that this is a weaker condition than $\#K_1 \le ch^{-2}$ as it is required in [25]. For a detailed discussion on this, we refer to [25, Lemma 4.7].

Definition 3.5. *Let T_h be a conforming triangulation of Ω. The projection R_h of a piecewise continuous function f is the piecewise constant function that fulfills*

$$R_h f \equiv f(S_T) \tag{3.6}$$

on any element $T \in T_h$. Here, S_T denotes the centroid of T.

In Section 4, we will prove a couple of properties of R_h, that allows us to formulate the following lemma.

Lemma 3.6. *Let T_h be a graded mesh according to* (3.1) *that satisfies condition* (3.5)*. Then the estimates*

$$\|S_h\bar{u} - S_h R_h \bar{u}\|_{L^2(\Omega)} \leq ch^2 \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})} \right), \tag{3.7}$$

$$\|P_h\bar{u} - P_h R_h \bar{u}\|_{L^2(\Omega)} \leq ch^2 \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})} \right) \tag{3.8}$$

are valid if $\mu < \lambda$.

The proof is postponed to Section 4. These estimates are the basis of the following supercloseness result. Originally, Meyer and Rösch discovered in [19] for isotropic and quasi-uniform grids, that the distance of the computed approximate solution $\bar{u}_h$ to the interpolant $R_h\bar{u}$ is much smaller than to the optimal solution $\bar{u}$ itself. Apel, Rösch and Winkler [6] and Apel and Winkler [8] extended this result to the case of isotropic, graded meshes in two and three dimensions. The next theorem shows, that this result transfers to the case of three-dimensional, anisotropic, graded meshes.

Theorem 3.7. *Let $\bar{u}_h$ be the solution of* (3.2)–(3.3) *on a family of meshes with grading parameter $\mu < \lambda$. Then the estimate*

$$\|\bar{u}_h - R_h\bar{u}\|_{L^2(\Omega)} \leq ch^2 \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})} \right)$$

holds true.

The proof is given in Section 5.

The final approximation is constructed from p_h by the pointwise projection into the set of admissible controls,

$$\tilde{u}_h = \Pi_{U^{\mathrm{ad}}} \left(-\frac{1}{\nu}\bar{p}_h \right). \tag{3.9}$$

Based on the supercloseness of $R_h\bar{u}$ to $\bar{u}_h$ given in Theorem 3.7, we can prove the following superconvergence result.

Theorem 3.8. *Let $\bar{u}$, $\bar{y}$, $\bar{p}$ and $\bar{u}_h$, $\bar{y}_h$, $\bar{p}_h$ be the solutions of* (1.7) *and* (3.4)*, respectively, where the family of meshes is graded with parameter $\mu < \lambda$ and satisfies condition* (3.5)*.*

Let $\tilde{u}_h$ be the postprocessed control constructed by (3.9)*. Then the estimates*

$$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \leq ch^2 \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})} \right) \tag{3.10}$$

$$\|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \leq ch^2 \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})} \right) \tag{3.11}$$

$$\|\bar{u} - \tilde{u}_h\|_{L^2(\Omega)} \leq ch^2 \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})} \right) \tag{3.12}$$

hold true.

Proof. The conclusion is similar to the one in [25, Subsection 4.8] except for the weaker condition on μ. For the sake of completeness we sketch it here. One has

$$\begin{aligned} \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} &= \|S\bar{u} - S_h\bar{u}_h\|_{L^2(\Omega)} \\ &\leq \|(S - S_h)\bar{u}\|_{L^2(\Omega)} + \|S_h(\bar{u} - R_h\bar{u})\|_{L^2(\Omega)} + \|S_h(R_h\bar{u} - \bar{u}_h\|_{L^2(\Omega)}. \end{aligned}$$

The application of Lemma 3.1, Lemma 3.6 and Theorem 3.7 yields together with the fact that S_h is a bounded operator from $L^2(\Omega)$ to $L^2(\Omega)$ (see Lemma 3.2) and the embedding $L^\infty(\Omega) \hookrightarrow L^2(\Omega)$ the assertion (3.10). For the second estimate one can write

$$\begin{aligned} \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} &= \|S^*(\bar{y} - y_d) - S_h^*(\bar{y}_h - y_d)\|_{L^2(\Omega)} \\ &\leq \|(S^* - S_h^*)(\bar{y} - y_d)\|_{L^2(\Omega)} + \|S_h^*(\bar{y} - \bar{y}_h)\|_{L^2(\Omega)}. \end{aligned}$$

The application of Lemma 3.1, (3.10) and Lemma 3.2 results in inequality (3.11). Finally estimate (3.12) follows directly from

$$\|\bar{u} - \tilde{u}_h\|_{L^2(\Omega)} = \left\| \Pi_{[a,b]} \left(-\frac{1}{\nu}\bar{p} \right) - \Pi_{[a,b]} \left(-\frac{1}{\nu}\bar{p}_h \right) \right\|_{L^2(\Omega)} \leq \frac{1}{\nu} \|\bar{p} - \bar{p}_h\|_{L^2(\Omega)}.$$

and estimate (3.11). □

Remark 3.9. *The first and the third author proved in [8] the result of Theorem 3.8 for domains with corner- and edge singularities and appropriately graded isotropic meshes. In detail, the mesh was chosen such that the condition*

$$\begin{aligned} h_T &\sim h^{1/\mu} \text{ for } r_T = 0, \\ h_T &\sim h r_T^{1-\mu} \text{ for } r_T > 0 \end{aligned}$$

is satisfied, where h_T denotes the diameter of the element T and r_T its distance to the set of singular points. The grading parameter μ had to fulfill the three conditions

$$\mu < \frac{1}{2} + \frac{1}{2}\lambda_v, \quad \mu < \lambda_e, \quad \mu < \frac{1}{3} + \frac{1}{2}\lambda_e. \tag{3.13}$$

Here λ_v and λ_e denote particular eigenvalues of certain operator pencils that correpond to the corner- and edge singularities, respectively. As in the case of anisotropic refinement, a weaker condition, namely

$$\mu < \min \left\{ \frac{1}{2} + \lambda_v, \lambda_e \right\} \tag{3.14}$$

is sufficient to get an optimal convergence rate for the boundary value problem [5]. Let us quickly describe where the additional conditions $\mu < \frac{1}{3} + \frac{1}{2}\lambda_e$ *and* $\mu < \frac{1}{2} + \frac{1}{2}\lambda_v$ *come from in [8]. In that paper the boundedness for* $r^\beta \nabla p$ *was proved for*

$$\beta > \max\left\{\frac{4}{3} - \lambda_e, 1 - \lambda_v\right\} \tag{3.15}$$

by the use of Sobolev embedding theorems. In the proof of Lemma 4.5 in [8] one needed the boundedness of $r^{2-2\mu}\nabla p$*. This resulted in the condition* $2-2\mu > \max\{\frac{4}{3} - \lambda_e, 1-\lambda_v\}$*, i.e.* $\mu < \min\{\frac{1}{3} + \frac{1}{2}\lambda_e, \frac{1}{2} + \frac{1}{2}\lambda_v\}$*. With an analogous argumentation as in Lemma 2.3 and Corollary 2.4 one can prove that* $r^\beta \nabla p$ *is already bounded for*

$$\beta > \max\{1 - \lambda_e, 1 - \lambda_v\}$$

as long as the desired state is in $C^{0,\sigma}(\bar{\Omega})$*. This means a smaller weight than stated in* (3.15) *is sufficient to compensate a possible edge singularity. Consequently, the condition in the proof of Lemma 4.5 in [8] reduces to* $2-2\mu > \max\{1-\lambda_e, 1-\lambda_v\}$*, what is fulfilled by values of* μ *that satisfy* $\mu < \min\{\frac{1}{2} + \frac{1}{2}\lambda_e, \frac{1}{2} + \frac{1}{2}\lambda_v\}$*. Since* $\lambda_e \leq 1$ *the condition* $\mu < \frac{1}{2} + \frac{1}{2}\lambda_e$ *is weaker than* $\mu < \lambda_e$*. Therefore one gets second order convergence on isotropic graded meshes already for a grading parameter* μ *satisfying*

$$\mu < \min\left\{\frac{1}{2} + \frac{1}{2}\lambda_v, \lambda_e\right\}$$

what is of course a weaker condition than the original condition (3.13)*. Notice, that this condition is still slightly stronger than condition* (3.14)*.*

The remainder of this paper contains the proofs of Lemma 3.6 (Sect. 4), Theorem 3.7 (Sect. 5) and a numerical test (Sect. 6).

4 Properties of the operator R_h

First of all we introduce the sets

$$K_s = \bigcup_{\{T \in T_h : r_T = 0\}} T \quad \text{and} \quad K_r = \Omega \backslash \bar{K}_s. \tag{4.1}$$

Notice, that according to (3.1) the number n of elements in K_s is $O(h^{-1})$ and therefore $|K_s| \leq cnh^{2/\mu+1} = ch^{2/\mu}$.

We collect here a number of results from [25] and refer for proofs to this thesis. We give proofs here only in those cases when changes are necessary due to the weaker mesh condition $\mu < \lambda$ in comparison with $\mu < \min\{\lambda, \frac{\lambda}{3} + \frac{5}{9}\}$ in [25], or when the proof in [25] is restricted to an analogy argument to a further result.

First of all, we recall the approximation properties of the operator R_h.

Lemma 4.1. *[25, Lemma 3.24] Let T_h be a conforming anisotropic triangulation satisfying equation* (3.1) *and let R_h be the projection defined in* (3.6)*. Then there holds*

$$\left|\int_T (f - R_h f)\,\mathrm{d}x\right| \leq \begin{cases} c|T|^{1/2} \sum_{|\alpha|=2} h_T^\alpha \|D^\alpha f\|_{L^2(T)} & \text{for } f \in H^2(T) \\ c|T| \sum_{|\alpha|=1} h_T^\alpha \|D^\alpha f\|_{L^\infty(T)} & \text{for } f \in W^{1,\infty}(T) \\ c|T| \|f\|_{L^\infty(T)} & \text{for } f \in L^\infty(T). \end{cases}$$

In the following we introduce the L^2-projection in the space of piecewise constant functions.

Definition 4.2. *Let T_h be a conforming triangulation of Ω. The L^2-projection of a function $f \in L^2(\Omega)$ is the piecewise constant function that fulfills*

$$Q_h f \equiv \frac{1}{|T|} \int_T f(x)\,\mathrm{d}x$$

on any element $T \in T_h$.

Lemma 4.3. *[25, Lemma 3.19] For any element $T \in \mathcal{T}_h$ and any function $f \in H^1(T)$ the inequality*

$$\|f - Q_h f\|_{L^2(T)} \leq ch|f|_{H^1(T)}$$

holds.

Corollary 4.4. *[25, Corollary 3.20] For any element $T \in \mathcal{T}_h$ and any two functions $f \in H^1(T)$, $v \in H^1(T)$ the inequality*

$$(f - Q_h f, v)_{L^2(T)} \leq ch^2 |f|_{H^1(T)} |v|_{H^1(T)}$$

is valid.

Lemma 4.5. *[25, Lemma 4.13] The inequality*

$$\|Q_h f - R_h f\|_{L^2(T)} \leq |T|^{1/2-1/p} \sum_{|\alpha|=1} h_T^\alpha \|D^\alpha f\|_{L^p(T)}$$

holds for all $f \in W^{1,p}(T)$ with $p > 3$.

Proof. By the definition of Q_h and R_h one has

$$\int_T (Q_h f - R_h f)^2\,\mathrm{d}x = \int_T \left[\frac{1}{|T|}\int_T f - R_h f \mathrm{d}\xi\right]^2 \mathrm{d}x = |T|^{-1}\left[\int_T f - R_h f \mathrm{d}\xi\right]^2$$

which leads to

$$\|Q_h f - R_h f\|_{L^2(T)} \leq |T|^{-1/2} \left|\int_T f - R_h f\,\mathrm{d}x\right|. \tag{4.2}$$

For any $\hat{w} \in \mathcal{P}_0(\hat{T})$ we can conclude

$$\begin{aligned}\int_T (f - R_h f)\,\mathrm{d}x &= |T| \int_{\hat{T}} (\hat{f} - \hat{R}\hat{f})\,\mathrm{d}x = |T| \int_{\hat{T}} (\hat{f} - \hat{w}) - \hat{R}(\hat{f} - \hat{w})\,\mathrm{d}x \\ &\leq c|T| \|\hat{f} - \hat{w}\|_{L^\infty(\hat{T})} \leq c|T| \|\hat{f} - \hat{w}\|_{W^{1,p}(\hat{T})}\end{aligned}$$

where we have used the embedding $L^\infty(\hat{T}) \hookrightarrow W^{1,p}(\hat{T})$ for $p > 3$. Now we can apply the Deny-Lions lemma and get

$$\int_T (f - R_h f)\,\mathrm{d}x \leq c|T| |\hat{f}|_{W^{1,p}(\hat{T})} \leq c|T|^{1-1/p} \sum_{|\alpha|=1} h_T^\alpha \|D^\alpha f\|_{L^p(T)}$$

which yields together with estimate (4.2) the assertion. □

Corollary 4.6. *[25, Corollary 4.16] Let the mesh be graded according to* (3.1)*. Then*

$$\|Q_h w - R_h w\|_{L^2(K_s)} \leq ch^2 \left(\|\partial_1 w\|_{L^p(K_s)} + \|\partial_2 w\|_{L^p(K_s)} + \|r^{-\mu} \partial_3 w\|_{L^p(K_s)} \right)$$

holds for all $w \in W^{1,p}(K_s)$ *with* $r^{-\mu}\partial_3 w \in L^p(K_s)$ *and* $p > 3$, $p \geq \frac{1}{1-\mu}$.

Corollary 4.7. *Let the mesh be graded according to* (3.1)*. Then*

$$\|Q_h w - R_h w\|_{L^2(K_r)} \leq ch^2 \left(|w|_{V^{2,2}_{2-2\mu}(K_r)} + |\partial_3 w|_{V^{2,1}_{1-\mu}(K_r)} + \|\partial_{33} w\|_{L^2(K_r)} \right)$$

holds for all $w \in H^2(K_r)$.

Proof. The proof is taken from [25, pages 48f. and 23] From the definition of Q_h one has

$$\|Q_h w - R_h w\|^2_{L^2(K_r)} = \sum_{T \subset K_r} \|Q_h w - R_h w\|^2_{L^2(T)} = \sum_{T \subset K_r} |T|^{-1} \left| \int_T (w - R_h w)\,\mathrm{d}x \right|^2$$

We apply Lemma 4.1 and get

$$\begin{aligned}\|Q_h w - R_h w\|^2_{L^2(K_r)} &\leq \sum_{T \subset K_r} |T|^{-1} \left[c \sum_{|\alpha|=2} h_T^\alpha |T|^{1/2} \|D^\alpha w\|_{L^2(T)} \right]^2 \\ &\leq c \sum_{T \subset K_r} \left[\sum_{|\alpha|=2} h_T^\alpha \|D^\alpha w\|_{L^2(T)} \right]^2.\end{aligned}$$

Since $r_T > 0$ for an element $T \subset K_r$ it follows with (3.1)

$$\begin{aligned}\|Q_h w - R_h w\|^2_{L^2(K_r)} \leq ch^4 \sum_{T \subset K_r} \Bigg[& r_T^{2-2\mu} \sum_{i=1}^2 \sum_{j=1}^2 \|\partial_{ij} y\|_{L^2(T)} + \\ & r_T^{1-\mu} \sum_{i=1}^2 \|\partial_{3i} y\|_{L^2(T)} + \|\partial_{33} y\|_{L^2(T)} \Bigg]^2 \\ \leq ch^4 \left(|w|_{V^{2,2}_{2-2\mu}(K_r)} + |\partial_3 w|_{V^{1,2}_{1-\mu}(K_r)} + \|\partial_{33} w\|_{L^2(K_r)} \right)^2.\end{aligned}$$

Extracting the root yields the assertion. □

The following lemma includes a stronger result in comparison with [25, Corollary 4.16].

Corollary 4.8. *Let the mesh be graded according to* (3.1)*. Then*

$$\|Q_h w - R_h w\|_{L^2(K_s)} \le ch^2 \|\nabla w\|_{L^\infty(K_s)}$$

holds for all $w \in W^{1,\infty}(K_s)$.

Proof. One can conclude from the Definition of Q_h and Lemma 4.1

$$\begin{aligned}
\|Q_h w - R_h w\|^2_{L^2(K_s)} &= \sum_{T\subset K_s} \int_T \left[|T|^{-1} \int_T w - R_h w \,\mathrm{d}\xi \right]^2 \mathrm{d}x \\
&= \sum_{T\subset K_s} |T|^{-1} \left[\int_T w - R_h w \,\mathrm{d}\xi \right]^2 \\
&\le c \sum_{T\subset K_s} |T| \left(\sum_{|\alpha|=1} h_T^\alpha \|D^\alpha w\|_{L^\infty(T)} \right)^2 .
\end{aligned}$$

If one takes into account that $\#K_s \le ch^{-1}$ it follows

$$\|Q_h w - R_h w\|^2_{L^2(K_s)} \le ch^{3+2/\mu} \#K_s \|\nabla w\|^2_{L^\infty(K_s)} \le ch^{2+2/\mu} \|\nabla w\|^2_{L^\infty(K_s)}.$$

Since $\mu \le 1$ this yields the assertion. □

Before we are able to show Lemma 3.6, we state one more auxiliary result. The proof of this result uses the boundedness of $r^\beta Pu$ for $\beta > 1 - \lambda$ stated in Corollary 2.4. In [25] this boundedness was only proven for $\beta > \frac{4}{3} - \lambda$. Our improvement allows us to weaken the grading condition from $\mu < \min\{\lambda, \frac{5}{9} + \frac{\lambda}{3}\}$ as it is given in [25] to $\mu < \lambda$. Notice, that the condition $\mu < \lambda$ was also necessary to get optimal convergence of the finite element approximation of the state equation (comp. [1, 7]).

Lemma 4.9. *Let* T_h *be an anisotropic, graded mesh satisfying* (3.1)*. Let* $\bar u$ *be the solution of the optimal control problem* (1.1)-(1.2)*. Then the estimate*

$$(Q_h \bar u - R_h \bar u, v_h)_{L^2(\Omega)} \le ch^2 \|v_h\|_{L^\infty(\Omega)} \left(\|\bar u\|_{L^\infty(\Omega)} + \|\bar y_d\|_{C^{0,\sigma}(\bar\Omega)} \right)$$

holds for all $v_h \in V_h$ *if* $\mu < \lambda$.

Proof. The proof follows the lines of that of Lemma 4.10 in [25]. Since the mesh grading condition is weakened from $\mu < \min\{\lambda, \frac{5}{9} + \frac{\lambda}{3}\}$ to $\mu < \lambda$, a detailed proof is given. We split the domain Ω into three parts, where $\bar u$ has different regularity: $K_{1,r} = K_1 \backslash \bar K_s$, $K_{2,r} = K_2 \backslash \bar K_s$ and K_s. One has

$$\int_\Omega v_h (Q_h \bar u - R_h \bar u) \,\mathrm{d}x \le \sum_{T\in\mathcal{T}_h} \|v_h\|_{L^\infty(T)} \int_T (\bar u - R_h \bar u) \,\mathrm{d}x.$$

If we apply Lemma 4.1 on each sub-domain to the integral, we get

$$\begin{aligned}\int_\Omega v_h(Q_h\bar{u}-R_h\bar{u})\,\mathrm{d}x \le &\sum_{T\subset K_{2,r}} \|v_h\|_{L^\infty(T)}|T|^{1/2}\sum_{|\alpha|=2} h_T^\alpha \|D^\alpha \bar{u}\|_{L^2(T)}\\ &+\sum_{T\subset K_{1,r}} \|v_h\|_{L^\infty(T)}|T|\sum_{|\alpha|=1} h_T^\alpha \|D^\alpha \bar{u}\|_{L^\infty(T)} \qquad (4.3)\\ &+\sum_{T\subset K_s} \|v_h\|_{L^\infty(T)}|T|\|\bar{u}\|_{L^\infty(T)}.\end{aligned}$$

We estimate the three terms on the right-hand side separately using (3.1). For the first term we have

$$\sum_{T\subset K_{2,r}} \|v_h\|_{L^\infty(T)}|T|^{1/2}\sum_{|\alpha|=2} h_T^\alpha \|D^\alpha \bar{u}\|_{L^2(T)} \qquad (4.4)$$

$$\begin{aligned}\le & c\|v_h\|_{L^\infty(K_{2,r})}|K_{2,r}|^{1/2}\left(h^2 r^{2-2\mu}\sum_{i=1}^2\sum_{j=1}^2 \|\partial_{ij}\bar{u}\|_{L^2(K_{2,r})}\right.\\ &\left.+\, h^2 r^{1-\mu}\sum_{i=1}^2 \|\partial_{3i}\bar{u}\|_{L^2(K_{2,r})}+h^2\|\partial_{33}\bar{u}\|_{L^2(K_{2,r})}\right). \qquad (4.5)\end{aligned}$$

The second term can be estimated by using (3.1),

$$\begin{aligned}&\sum_{T\subset K_{1,r}} \|v_h\|_{L^\infty(T)}|T|\sum_{|\alpha|=1} h_T^\alpha \|D^\alpha \bar{u}\|_{L^\infty(T)}\\ &\quad\le ch\|v_h\|_{L^\infty(\Omega)}\sum_{T\subset K_{1,r}}|T|\left(\sum_{i=1}^2 \|r^{1-\mu}\partial_i\bar{u}\|_{L^\infty(T)}+\|\partial_3\bar{u}\|_{L^\infty(T)}\right)\\ &\quad\le ch\|v_h\|_{L^\infty(\Omega)}|K_{1,r}|\left(\sum_{i=1}^2 \|r^{1-\mu}\partial_i\bar{u}\|_{L^\infty(K_{1,r})}+\|\partial_3\bar{u}\|_{L^\infty(K_{1,r})}\right)\\ &\quad\le ch^2\|v_h\|_{L^\infty(\Omega)}\left(\sum_{i=1}^2 \|r^{1-\mu}\partial_i\bar{u}\|_{L^\infty(K_{1,r})}+\|\partial_3\bar{u}\|_{L^\infty(K_{1,r})}\right). \qquad (4.6)\end{aligned}$$

The last step is valid since $|K_1|\le ch$ (comp. (3.5)). The third term yields

$$\sum_{T\subset K_s} \|v_h\|_{L^\infty(T)}|T|\|\bar{u}\|_{L^\infty(T)} \le |K_s|\|v_h\|_{L^\infty(\Omega)}\|\bar{u}\|_{L^\infty(K_s)} \le ch^2\|v_h\|_{L^\infty(\Omega)}\|\bar{u}\|_{L^\infty(K_s)} \qquad (4.7)$$

since $|K_s|\le ch^{2/\mu}\le ch^2$. We can further utilize the projection formula (1.8) and substitute $\bar{u}$ by $-\frac{1}{\nu}\bar{p}$ in the above norms, because $\bar{u}$ is either constant or equal to $-\frac{1}{\nu}p$.

Then the inequalities (4.5), (4.6) and (4.7) yield together with (4.3) the estimate

$$\int_\Omega v_h(Q_h\bar{u} - R_h\bar{u})\,\mathrm{d}x \le \frac{c}{\nu}h^2\|v_h\|_{L^\infty(\Omega)}\cdot$$

$$\left(\sum_{i=1}^{2}\sum_{j=1}^{2}\|r^{2-2\mu}\partial_{ij}\bar{p}\|_{L^2(K_{2,r})} + \sum_{i=1}^{2}\|r^{1-\mu}\partial_{3i}\bar{p}\|_{L^2(K_{2,r})} + \|\partial_{33}\bar{p}\|_{L^2(K_{2,r})} \right. \tag{4.8}$$

$$\left. + \sum_{i=1}^{2}\|r^{1-\mu}\partial_i\bar{p}\|_{L^\infty(K_{1,r})} + \|\partial_3\bar{p}\|_{L^\infty(K_{1,r})} + \nu\|\bar{u}\|_{L^\infty(K_s)}\right). \tag{4.9}$$

In order to estimate the L^2-norms in (4.8), we split $\bar{p}$ according to (2.2) in a regular and a singular part, $\bar{p} = \bar{p}_r + \bar{p}_s$. Then we apply Lemma 2.1 for $p = 2$ with $\beta = 2 - 2\mu$. This is possible since $\mu < \lambda < 1$ and therefore $2 - 2\mu > 2 - \lambda - 1 = 1 - \lambda$. Notice that one has for the regular part $\|r^\alpha\bar{p}_r\|_{H^2(\Omega)} \le c\|\bar{p}\|_{H^2(\Omega)}$ as long as $\alpha > 0$. For the estimate of the L^∞-norms in (4.9), we apply Corollary 2.4. We end up with

$$\int_\Omega v_h(Q_h\bar{u} - R_h\bar{u})\,\mathrm{d}x \le ch^2\|v_h\|_{L^\infty(\Omega)}\left(\|y - y_d\|_{L^2(\Omega)} + \|y - y_d\|_{C^{0,\sigma}(\bar{\Omega})} + \|u\|_{L^\infty(\Omega)}\right)$$

for a $\sigma \in (0,1)$. If we use the triangle inequality, the embedding $L^\infty(\Omega) \hookrightarrow L^2(\Omega)$ and estimate (2.20) the assertion is shown. □

Proof of Lemma 3.6. The proof is similiar to that of Lemma 4.19 in [25], where it is given for $\mu < \min\{\lambda, \frac{5}{9} + \frac{\lambda}{3}\}$. We recall the proof here under the weaker condition $\mu < \lambda$. By the definiton of P_h and S_h one has

$$\begin{aligned}
\|S_h\bar{u} - S_hR_h\bar{u}\|^2_{L^2(\Omega)} &= (S_h\bar{u} - S_hR_h\bar{u}, S_h\bar{u} - S_hR_h\bar{u})_{L^2(\Omega)}\\
&= a(S_h\bar{u} - S_hR_h\bar{u}, P_h\bar{u} - P_hR_h\bar{u})\\
&= (\bar{u} - R_h\bar{u}, P_h\bar{u} - P_hR_h\bar{u})_{L^2(\Omega)}\\
&= (\bar{u} - Q_h\bar{u}, P_h\bar{u} - P_hR_h\bar{u})_{L^2(\Omega)} + (Q_h\bar{u} - R_h\bar{u}, P_h\bar{u} - P_hR_h\bar{u})_{L^2(\Omega)}.
\end{aligned} \tag{4.10}$$

where we have used

$$P_h\bar{u} - P_hR_h\bar{u} = S_h^*(S_h\bar{u} - y_d) - S_h^*(S_hR_h\bar{u} - y_d) = S_h^*(S_h\bar{u} - S_hR_h\bar{u}). \tag{4.11}$$

We estimate the two terms seperately. For the first term we have with Corollary 4.4

$$\begin{aligned}
\sum_{T\in\mathcal{T}_h}(\bar{u} - Q_h\bar{u}, P_h\bar{u} - P_hR_h\bar{u})_{L^2(T)} &\le c\sum_{T\in\mathcal{T}_h}h^2|\bar{u}|_{H^1(T)}|P_h\bar{u} - P_hR_h\bar{u}|_{H^1(T)}\\
&\le ch^2|\bar{u}|_{H^1(\Omega)}|P_h\bar{u} - P_hR_h\bar{u}|_{H^1(\Omega)}
\end{aligned}$$

because $h_T^2 \le ch^2$. With (4.11) and Lemma 3.2 we can continue with

$$\begin{aligned}
(\bar{u} - Q_h\bar{u}, P_h\bar{u} - P_hR_h\bar{u})_{L^2(\Omega)} &\le ch^2|\bar{u}|_{H^1(\Omega)}\|S_h^*\|_{L^2(\Omega)\to H^1(\Omega)}\|S_h\bar{u} - S_hR_h\bar{u}\|_{L^2(\Omega)}\\
&\le ch^2|\bar{u}|_{H^1(\Omega)}\|S_h\bar{u} - S_hR_h\bar{u}\|_{L^2(\Omega)}.
\end{aligned}$$

According to the projection formula (1.8) the optimal control $\bar{u}$ is either constant or equal to $-\frac{1}{\nu}p$. Therefore one can conclude from the boundedness of S^* from $L^2(\Omega)$ in $H^1(\Omega)$

$$|\bar{u}|_{H^1(\Omega)} \le c\|\bar{p}\|_{H^1(\Omega)} = c\|S^*(\bar{y}-y_d)\|_{H^1(\Omega)} \le c\|\bar{y}-y_d\|_{L^2(\Omega)} \le c\left(\|\bar{y}\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)}\right).$$

This yields

$$(\bar{u}-Q_h\bar{u}, P_h\bar{u}-P_hR_h\bar{u})_{L^2(\Omega)} \le ch^2\left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right)\|S_h\bar{u}-S_hR_h\bar{u}\|_{L^2(\Omega)}. \tag{4.12}$$

We continue with the second term of (4.10). The application of Lemma 4.9, Equation (4.11) and again the boundedness of S_h^* yields

$$\begin{aligned}(Q_h\bar{u}-R_h\bar{u}, P_h\bar{u}-P_hR_h\bar{u})_{L^2(\Omega)} &\le ch^2\left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right)\|P_h\bar{u}-P_hR_h\bar{u}\|_{L^\infty(\Omega)}\\ &\le ch^2\left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right)\|S_h\bar{u}-S_hR_h\bar{u}\|_{L^2(\Omega)}\end{aligned}$$

This estimate gives together with (4.12) and after division by $\|S_h\bar{u}-S_hR_h\bar{u}\|_{L^2(\Omega)}$ the assertion (3.7). The inequality (3.8) follows from the estimate (3.7), equation (4.11) and the boundedness of S_h^* as operator from $L^2(\Omega)$ to $L^2(\Omega)$. □

5 Proof of the supercloseness of $R_h\bar{u}_h$ and $\bar{u}_h$

Before we prove a supercloseness result we recall the following lemma from [6], where it is proved for plane domains and isotropic graded meshes. The proof is valid also in the three-dimensional situation with anisotropic meshes.

Lemma 5.1. *[6, Lemma 7] The inequality*

$$\nu\|R_h\bar{u}-\bar{u}_h\|^2_{L^2(\Omega)} \le (R_h\bar{p}-\bar{p}_h, \bar{u}_h-R_h\bar{u})_{L^2(\Omega)}$$

is valid.

Proof of Theorem 3.7. This theorem is proved in [25] for the stronger mesh grading condition $\mu < \min\{\lambda, \frac{5}{9}+\frac{\lambda}{3}\}$ and for a special type of mixed boundary conditions. We give the proof here under the weaker condition $\mu < \lambda$ and for pure Dirichlet and Neumann boundary conditions. From Lemma 5.1 we have

$$\begin{aligned}\nu\|R_h\bar{u}-\bar{u}_h\|^2_{L^2(\Omega)} &\le (R_h\bar{p}-\bar{p}_h, \bar{u}_h-R_h\bar{u})_{L^2(\Omega)}\\ &= (R_h\bar{p}-\bar{p}, \bar{u}_h-R_h\bar{u})_{L^2(\Omega)} + (\bar{p}-P_hR_h\bar{u}, \bar{u}_h-R_h\bar{u})_{L^2(\Omega)}\\ &\quad + (P_hR_h\bar{u}-\bar{p}_h, \bar{u}_h-R_h\bar{u})_{L^2(\Omega)}\end{aligned} \tag{5.1}$$

Now, we estimate the three terms separately. For the first one can conclude

$$\begin{aligned}(R_h\bar{p}-\bar{p}, \bar{u}_h-R_h\bar{u})_{L^2(\Omega)} &= (R_h\bar{p}-Q_h\bar{p}, \bar{u}_h-R_h\bar{u})_{L^2(\Omega)} + (Q_h\bar{p}-\bar{p}, \bar{u}_h-R_h\bar{u})_{L^2(\Omega)}\\ &= (R_h\bar{p}-Q_h\bar{p}, \bar{u}_h-R_h\bar{u})_{L^2(\Omega)}\\ &\le \|R_h\bar{p}-Q_h\bar{p}\|_{L^2(\Omega)}\|\bar{u}_h-R_h\bar{u}\|_{L^2(\Omega)}\end{aligned} \tag{5.2}$$

where we have used the projection property of Q_h and the Cauchy-Schwarz inequality. In order to estimate the first term of the right-hand side of (5.2) we write

$$\|R_h\bar{p} - Q_h\bar{p}\|^2_{L^2(\Omega)} = \|R_h\bar{p} - Q_h\bar{p}\|^2_{L^2(K_s)} + \|R_h\bar{p} - Q_h\bar{p}\|^2_{L^2(K_r)}. \tag{5.3}$$

with K_s and K_r as defined in (4.1). In the following we choose p such that $p > 3$ and $p \geq \frac{1}{1-\mu}$. According to (2.2) we split $\bar{p}$ in a singular part $\bar{p}_s \in V^{2,p}_\beta(K_s)$ and a regular part $\bar{p}_r \in W^{2,p}(K_s)$ such that $\bar{p} = \bar{p}_s + \bar{p}_r$. For the singular part we get from (5.3) and the Corollaries 4.6 and 4.7 the estimate

$$\begin{aligned}\|R_h\bar{p}_s - Q_h\bar{p}_s\|_{L^2(\Omega)} \leq ch^2 \Big(&|\bar{p}_s|_{V^{2,2}_{2-2\mu}(K_r)} + |\partial_3\bar{p}_s|_{V^{1,2}_{1-\mu}(K_r)} + |\partial_{33}\bar{p}_s|_{V^{0,2}_0(K_r)} \\ &+ \|\partial_1\bar{p}_s\|_{L^p(K_s)} + \|\partial_2\bar{p}_s\|_{L^p(K_s)} + \|\partial_3\bar{p}_s\|_{V^{0,p}_{-\mu}(K_s)}\Big) \\ &\leq ch^2\|\bar{y} - y_d\|_{L^p(\Omega)}\end{aligned} \tag{5.4}$$

where we have used the estimates (2.3)–(2.5) in the last step. Since $W^{2,p}(\Omega) \hookrightarrow H^2(\Omega)$ and $W^{2,p}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ it follows from (5.3) and the Corollaries 4.7 and 4.8

$$\begin{aligned}&\|R_h\bar{p}_r - Q_h\bar{p}_r\|_{L^2(\Omega)} \\ &\quad\leq ch^2 \left(|\bar{p}_r|_{V^{2,2}_{2-2\mu}(K_r)} + |\partial_3\bar{p}_r|_{V^{1,2}_{1-\mu}(K_r)} + |\partial_{33}\bar{p}_r|_{V^{0,2}_0(K_r)} + \|\nabla\bar{p}_r\|_{L^\infty(K_s)}\right) \\ &\quad\leq ch^2\|\bar{y} - y_d\|_{L^p(\Omega)}\end{aligned} \tag{5.5}$$

where we have used the estimate (2.6) in the last step. Since

$$\|R_h\bar{p} - Q_h\bar{p}\|_{L^2(\Omega)} \leq \|R_h\bar{p}_s - Q_h\bar{p}_s\|_{L^2(\Omega)} + \|R_h\bar{p}_r - Q_h\bar{p}_r\|_{L^2(\Omega)}$$

one can conclude from (5.4) and (5.5)

$$\|R_h\bar{p} - Q_h\bar{p}\|_{L^2(\Omega)} \leq ch^2\|\bar{y} - y_d\|_{L^p(\Omega)}$$

Finally it follows from the triangle inequality and Lemma 2.1

$$\|R_h\bar{p} - Q_h\bar{p}\|_{L^2(\Omega)} \leq ch^2 \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right).$$

This yields together with (5.2) the estimate

$$(R_h\bar{p} - \bar{p}, \bar{u}_h - R_h\bar{u})_{L^2(\Omega)} \leq ch^2\|\bar{u}_h - R_h\bar{u}\|_{L^2(\Omega)} \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)}\right). \tag{5.6}$$

For the second term of (5.1) we get

$$(\bar{p} - P_hR_h\bar{u}, \bar{u}_h - R_h\bar{u})_{L^2(\Omega)} \leq \|\bar{p} - P_hR_h\bar{u}\|_{L^2(\Omega)}\|\bar{u}_h - R_h\bar{u}\|_{L^2(\Omega)}.$$

The triangle inequality yields together with (3.8) and Lemma (3.3)

$$\begin{aligned}\|\bar{p} - P_hR_h\bar{u}\|_{L^2(\Omega)} &\leq \|P\bar{u} - P_h\bar{u}\|_{L^2(\Omega)} + \|P_h\bar{u} - P_hR_h\bar{u}\|_{L^2(\Omega)} \\ &\leq ch^2 \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right)\end{aligned}$$

and therefore

$$(\bar{p} - P_h R_h \bar{u}, \bar{u}_h - R_h \bar{u})_{L^2(\Omega)} \leq ch^2 \|\bar{u}_h - R_h \bar{u}\|_{L^2(\Omega)} \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})} \right). \quad (5.7)$$

The third term of inequality (5.1) can simply be omitted since

$$\begin{aligned}(P_h R_h \bar{u} - \bar{p}_h, \bar{u}_h - R_h \bar{u})_{L^2(\Omega)} &= (S_h^* S_h R_h \bar{u} - S_h^* S_h \bar{u}_h, \bar{u}_h - R_h \bar{u})_{L^2(\Omega)} \\ &= (S_h R_h \bar{u} - S_h \bar{u}_h, S_h \bar{u}_h - S_h R_h \bar{u})_{L^2(\Omega)} \\ &= -\|S_h \bar{u} - S_h R_h \bar{u}\|_{L^2(\Omega)} \leq 0. \end{aligned} \quad (5.8)$$

Thus the inequalities (5.1), (5.6), (5.7) and (5.8) yield the assertion after dividing by $\|\bar{u}_h - R_h \bar{u}\|_{L^2(\Omega)}$. □

6 Numerical test

In this section we illustrate our theoretical findings by a numerical example. We consider the optimal control problem (1.1)–(1.2) with the state equation

$$-\Delta y = u + f \ \text{ in } \Omega, \qquad y = 0 \ \text{ on } \partial\Omega.$$

The domain Ω is chosen as

$$\Omega = \{(r\cos\varphi, r\sin\varphi, z) \in \mathbb{R}^3 : 0 < r < 1, 0 < \varphi < \omega_0, 0 < z < 1\}.$$

The functions f and y_d are defined such that

$$\begin{aligned}\bar{y}(r,\varphi,z) &= z(1-z)(r^\lambda - r^\alpha)\sin\lambda\varphi, \\ \bar{p}(r,\varphi,z) &= \nu z(1-z)(r^\lambda - r^\alpha)\sin\lambda\varphi \\ \bar{u}(r,\varphi,z) &= \Pi_{[-0.2,10.0]}\left(-\frac{1}{\nu}\bar{p}\right)\end{aligned}$$

is the exact solutions of the optimal control problem. We set $\omega_0 = \frac{11}{6}\pi$, $\nu = 10^{-3}$ and $\alpha = \frac{5}{2}$. Furthermore, we have $\lambda = \frac{\pi}{\omega} = \frac{6}{11}$.

The approximation is computed using an implementation of the primal-dual active set strategy by the third author. For a detailed description, we refer to [25, Sect. 5].

In Table 1 one can find the values for the error $\|\bar{u} - \tilde{u}_h\|_{L^2(\Omega)}$ as well as the estimated rate of convergence for different numbers of degrees of freedom. On the uniform mesh ($\mu = 1$) the convergence rate is significantly less than two, but larger than the rate of $2\lambda = 12/11$ as expected from the theory. However this is an asymptotic result for a region near the edge. In the case of an anisotropic graded mesh with $\mu = 0.54 < 6/11 = \lambda$, one can observe the predicted convergence rate of two.

	$\mu = 0.54$		$\mu = 1$	
ndof	value	rate	value	rate
546	1.48e−01		1.29e−01	
2310	5.16e−02	2.20	4.66e−02	2.12
6090	2.62e−02	2.09	2.55e−02	1.87
12654	1.62e−02	1.98	1.70e−02	1.66
22770	1.09e−02	2.01	1.23e−02	1.65
37206	7.84e−03	2.03	9.39e−03	1.63
56730	5.86e−03	2.06	7.50e−03	1.59
82110	4.58e−03	2.01	6.18e−03	1.56
114114	3.68e−03	1.98	5.23e−03	1.53
153510	3.02e−03	1.99	4.50e−03	1.51
201066	2.51e−03	2.07	3.94e−03	1.47
257550	2.14e−03	1.95	3.50e−03	1.45
323730	1.85e−03	1.92	3.13e−03	1.44
400374	1.60e−03	1.98	2.84e−03	1.41

Table 1: L^2-error of the computed control $\tilde{u}_h$ for interior angle $\omega_0 = \frac{11}{6}\pi$.

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