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L^∞-error estimates on graded meshes with application to optimal control

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Abstract An L^∞-error estimate of the finite element approximation of an elliptic boundary value problem with Dirichlet boundary conditions in domains with corners is given. To achieve an approximation rate of $h^2|\ln h|$ the mesh has to be appropriately graded near corners with an interior angle larger than ω_0, with $\omega_0 = \frac{\pi}{2}$ for the Poisson problem. In contrast to previous publications the norm of the function, that has to be approximated, is separated from the constants in this estimate.

This result is applied to a linear-quadratic optimal control problem with constraints on the control. Two approaches are considered, one where the control is approximated by piecewise constant functions and improved by a post-processing step, the other where the control is not discretized. For both approaches a convergence rate of $h^2|\ln h|$ in the maximum norm is shown.

Key Words Linear-quadratic optimal control problems, control constraints, corner singularities, finite element method, error estimates, superconvergence.

AMS subject classification 49M25, 65N30

1 Introduction

Many results concerning L^∞-estimates of the finite element error of linear elliptic boundary value problems were published in the 1970's (see e.g. [5, 14, 20, 21, 23]). Such an estimate is a main ingredient of an L^∞-error estimate for a finite element discretization of an optimal control problem. But all of the results have in common that they are not suitable for our setting due to a restriction on quasi-uniform meshes, strong regularity

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assumption on the solution or the domain or missing exact regularity assumptions on the right-hand side. Therefore we need to extend these results.

Let us first introduce the optimal control problem. We consider

$$J(\bar{u}) = \min_{u \in U_{\text{ad}}} J(u), \tag{1.1}$$

$$J(u) := F(Su, u), \tag{1.2}$$

$$F(y, u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)}, \tag{1.3}$$

where the associated state $y = Su$ to the control u is the weak solution of the state equation

$$Ly = u \quad \text{in } \Omega, \qquad y = 0 \quad \text{on } \Gamma = \partial\Omega. \tag{1.4}$$

The operator L is defined as

$$Ly := -\nabla \cdot A(x)\nabla y + b(x) \cdot \nabla y + c(x)y \tag{1.5}$$

where $A \in C^\infty(\bar{\Omega}, \mathbb{R}^{2\times 2})$, $b \in C^\infty(\bar{\Omega}, \mathbb{R}^2)$ and $c \in C^\infty(\bar{\Omega})$. Further the coefficients are assumed to satisfy the conditions

$$m_0|\xi|^2 \le \xi^T A(x)\xi \qquad \forall(\xi, x) \in \mathbb{R}^2 \times \bar{\Omega},\ m_0 > 0$$

and

$$c(x) - \frac{1}{2}\nabla \cdot b(x) \ge 0 \qquad \forall x \in \Omega$$

ensuring ellipticity and coercivity, respectively. The control variable is constrained by

$$u_a \le u(x) \le u_b \qquad \text{for a.a. } x \in \Omega. \tag{1.6}$$

In this setting $y_d \in C^{0,\sigma}(\bar{\Omega})$ with $\sigma \in (0, 1]$ is the desired state, u_a and u_b are real numbers, and the regularization parameter $\nu > 0$ is a fixed positive number. Furthermore, $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain and the set of admissible controls is

$$U_{\text{ad}} := \{u \in L^2(\Omega) : u_a \le u \le u_b \ \text{ a.e. in } \Omega\}.$$

We further introduce the adjoint problem

$$L^*p = y - y_d \quad \text{in } \Omega, \qquad p = 0 \quad \text{on } \Gamma$$

where

$$L^*p := -\nabla \cdot A(x)\nabla p - b(x) \cdot \nabla p + c(x)p.$$

We denote by S^* the solution operator of this problem, that means we have

$$p = S^*(y - y_d).$$

Since one can also write

$$p = S^*(Su - y_d) = Pu$$

with an affine operator P the solution $p = Pu$ is called the associated adjoint state to u.

We discretize the optimal control problem by a finite element method. The state y and the adjoint state p are approximated by piecewise linear functions. For discretizing the control mainly two different approaches exist. One method is to use a piecewise constant approximation for the control with an additional post-processing step. Meyer and Rösch originally used this in [12]. They derived optimal L^2-error estimates on quasi-uniform meshes in convex domains. They also proved the suboptimal convergence rate 1 for the error in the L^∞-norm. This is the same result they got in [13] for a piecewise linear approximation. Apel, Rösch and Winkler gave in [1] an L^2-estimate for problems in non-convex domains, where they used the postprocessing technique on appropriately graded meshes. The other discretization concept was introduced by Hinze in [6]. In this approach of variational discretization the space of admissible controls is not discretized. Instead, the first order optimality condition and the discretization of the state and the adjoint state are used to derive the approximate control. An L^2-error estimate was shown on convex domains with quasi-uniform meshes. Further, the L^∞-error was estimated subject to the finite element error.

In this paper here, we will prove that for problem (1.1)–(1.6) the approximation error in the L^∞-Norm behaves in both approaches like $O(h^2|\ln h|)$. Notice, that this optimal estimate is new for the discrete approach even in the case of convex domains.

Let us mention, that independent of the choice of the discretization the finite element error in the state and the adjoint state plays an important role in the error analysis of the optimal control problem. Therefore we recall some results concerning pointwise error estimates for elliptic boundary value problems from the literature. Scott proved in [23] a convergence rate of $h^2|\ln h|$ for $L = -\Delta + 1$ and Neumann boundary conditions. This result is valid if $y \in W^{2,\infty}(\Omega)$ and if the mesh is quasi-uniform. Frehse and Rannacher considered in [5] the Dirichlet problem for the operator $L = -\nabla \cdot A\nabla$ in domains Ω with $\partial\Omega \in C^{2,\alpha}$ and for a discretization with quasi-uniform meshes. For $y \in W^{2,\infty}(\Omega)$ they got the convergence rate $h^2|\ln h|$, for a right-hand side $f \in L^\infty$ they proved the approximation order $h^2|\ln h|^2$. Since we consider a domain with corners, the boundary is not in $C^{2,\alpha}$ and the state y is in general not in $W^{2,\infty}(\Omega)$. So these results are not applicable. In [20] Schatz and Wahlbin derived pointwise estimates for the Poisson equation in domains with corners. In [21] they specified a refinement rule for the mesh in order to get the estimate

$$\|y - y_h\|_{L^\infty(\Omega)} \le ch^{2-\epsilon},$$

where y_h is the finite element solution using piecewise linear ansatz functions. The drawback of this result is the fact, that the error constant is not separated from a norm of the right-hand side of the boundary value problem. Especially it is not clear, what regularity has to be assumed for the right-hand side, since Schatz and Wahlbin only demand a "smooth" right-hand side. We would like to emphasize that in our case the right-hand side is the unknown control and therefore one cannot assume arbitrary smoothness. To circumvent this problem, we extend these results and show that the weak solution of

$$Ly = f \quad \text{in } \Omega, \qquad y = 0 \quad \text{on } \Gamma = \partial\Omega,$$

for $f \in C^{0,\sigma}(\bar{\Omega})$, $\sigma \in (0, 1]$ fulfills the inequality

$$\|y - y_h\|_{L^\infty(\Omega)} \le ch^2 |\ln h| \|f\|_{C^{0,\sigma}(\bar{\Omega})}. \tag{1.7}$$

provided the mesh is appropriately graded. Notice, that the mesh grading is necessary for angles larger than a critical angle $\omega_0 < \pi$. For the Laplace operator one has $\omega_0 = \frac{\pi}{2}$. For mixed boundary conditions, that we have not considered here, ω_0 is even smaller than $\frac{\pi}{2}$. In order to extend the results of this paper to the case of mixed boundary conditions, one needs a regularity result analogous to Theorem 2.2. Further one has to ensure, that local estimates of the H^1-error given in Lemma 2.12 hold.

The outline of the paper is as follows. In Section 2 we start with giving some results concerning the regularity of the state equation. However, the main part of this section is the proof of the error estimate (1.7). In Section 3 we apply this result to the optimal control problem (1.1)–(1.6) and show that the approximation error in the control behaves for both discretization concepts mentioned above like $O(h^2|\ln h|)$. The paper is completed by numerical examples in Section 4.

2 The state equation

In this section, we derive an L^∞-error estimate for the finite element discretization of the state equation

$$Ly = f \quad \text{in } \Omega, \qquad y = 0 \quad \text{on } \partial\Omega. \tag{2.1}$$

with right-hand side $f \in C^{0,\sigma}(\bar{\Omega})$. First, we give some regularity results.

2.1 Regularity

It is well known, that the regularity of the solution of (2.1) is in general limited due to the corners of the domain Ω. Since this behavior is of local nature, we reduce our considerations for simplicity to one corner with interior angle $\omega > \omega_0$ located at the origin. The critical angle ω_0 is introduced below. We denote by

$$r := \sqrt{x_1^2 + x_2^2}$$

the distance to this corner. In order to describe the regularity of y we define for $k \in \mathbb{N}_0$ and $\beta \in \mathbb{R}$ the weighted Sobolev spaces

$$V_\beta^{k,p}(\Omega) = \left\{ v \in \mathcal{D}'(\Omega) \ : \ \|v\|_{V_\beta^{k,p}(\Omega)} < \infty \right\}$$

where

$$\|v\|_{V_\beta^{k,p}(\Omega)} := \left(\sum_{|\alpha|\le k}\int_\Omega |r^{\beta-k+|\alpha|}D^\alpha v|^p\right)^{1/p}$$

for $p<\infty$ and

$$\|v\|_{V_\beta^{k,\infty}(\Omega)} := \operatorname{ess\,sup}_{|\alpha|\le k,\ x\in\Omega} |r^{\beta-k+|\alpha|}D^\alpha v(x)|.$$

Notice, that the regularity of the solution y of the boundary value problem (2.1) is characterized by one particular eigenvalue $\lambda \ge \frac{1}{2}$ of an operator pencil, which is the result of an integral transformation, see [18]. For the Laplace operator the eigenvalue λ is explicitly known, $\lambda = \frac{\pi}{\omega}$. For the more general operator L defined in (1.5) one can compute λ by a linear coordinate transformation. We denote by ω_0 the angle for which $\lambda = 2$. This means that $\lambda < 2$ for $\omega > \omega_0$. For the Laplace operator one has $\omega_0 = \frac{\pi}{2}$. For more details we refer to [15, Chap. 5].

Lemma 2.1. *The embeddings*

$$V_\beta^{2,2}(\Omega) \hookrightarrow V_\gamma^{2,2}(\Omega) \quad \text{for } \beta<\gamma, \tag{2.2}$$

$$V_\gamma^{2,\infty}(\Omega) \hookrightarrow L^\infty(\Omega) \quad \text{for } \gamma\le 2 \tag{2.3}$$

hold.

Proof. Since $\beta<\gamma$ the embedding (2.2) follows directly from the definition of the spaces. For $u\in V_\gamma^{2,\infty}(\Omega)$ one has $r^{\gamma-2}u\in L^\infty(\Omega)$. From the fact that $\gamma\le 2$ one obtains $u\in L^\infty(\Omega)$ what proves (2.3). □

Theorem 2.2. *Let $\gamma > 2-\lambda \ge 0$ and $\sigma\in(0,1)$. Then the weak solution u of (2.1) belongs to the space $V_\gamma^{2,\infty}(\Omega)$, and the inequality*

$$\|y\|_{V_\gamma^{2,\infty}(\Omega)} \le c\|f\|_{C^{0,\sigma}(\bar\Omega)}$$

is valid.

Proof. In [10], the weighted Hölder spaces $N_\beta^{l,\sigma}(\Omega)$ are introduced with the norm

$$\|y\|_{N_\beta^{l,\sigma}(\Omega)} = \sup_{x\in\Omega}\sum_{|\alpha|\le l} r^{\beta-l-\sigma+|\alpha|}|D^\alpha y| + \sum_{|\alpha|=l}\sup_{x,x'\in\Omega}\frac{|(r^\beta D^\alpha y)(x)-(r^\beta D^\alpha y)(x')|}{|x-x'|^\sigma}.$$

In Section 8.7.1 of [10] it is shown, that for $-\lambda\le l+\sigma-\beta\le\lambda$ the regularity result

$$\|y\|_{N_\beta^{l,\sigma}(\Omega)} \le c\|f\|_{N_\beta^{l-2,\sigma}(\Omega)}$$

for the weak solution y of (2.1) holds. This means that in the case of $l=2$ one can conclude for $\gamma := \beta-\sigma > 2-\lambda$

$$r^{\gamma-2+|\alpha|}|D^\alpha y| \le c\|f\|_{N^{0,\sigma}_{\gamma+\sigma}(\Omega)} \quad \forall\alpha : |\alpha|\le 2.$$

and therefore

$$\|y\|_{V_\gamma^{2,\infty}(\Omega)} \le c\|f\|_{N_\beta^{0,\sigma}(\Omega)}.$$

According to [11, §5] the $N_\beta^{l,\sigma}(\Omega)$-norm is equivalent to

$$\sup_{\substack{x,x'\in\Omega\\ 2|x-x'|\le\min\{|x|,|x'|\}}} r(x)^\beta \sum_{|\alpha|=l} \frac{|D^\alpha y(x) - D^\alpha y(x')|}{|x-x'|^\sigma} + \sup_{x\in\Omega} r(x)^{\beta-l-\sigma}|y(x)|.$$

If one sets $l = 0$ this implies with$\gamma = \beta - \sigma \ge 0$ the embedding $C^{0,\sigma}(\bar\Omega) \hookrightarrow N_\beta^{0,\sigma}(\Omega)$ and the assertion is shown. □

2.2 Finite element error estimates

We will now discretize the boundary value problem (2.1). To this aim we introduce a family of graded triangulations $(T_h)_{h>0}$ of $\bar\Omega$. With a global mesh parameter h, a grading parameter $\mu \in (0,1]$ and the distance r_T of a triangle T to the corner,

$$r_T := \inf_{(x_1,x_2)\in T} \sqrt{x_1^2 + x_2^2},$$

we assume that the element size $h_T := \mathrm{diam} T$ satisfies

$$\begin{aligned} c_1 h^{1/\mu} \le h_T \le c_2 h^{1/\mu} \qquad &\text{for } r_T = 0\\ c_1 h r^{1-\mu} \le h_T \le c_2 h r^{1-\mu} \qquad &\text{for } r_T > 0. \end{aligned} \tag{2.4}$$

Notice, that the number of elements of such a triangulation is of order h^{-2}, see e.g. [2]. Finally, set V_h as the space of all piecewise linear and globally continuous functions in Ω,

$$V_h = \{v_h \in C(\bar\Omega) : v_h|_T \in \mathcal{P}_1 \text{ for all } T \in T_h \text{ and } v_h = 0 \text{ on } \partial\Omega\}.$$

The variational solution of (2.1) is given by the unique element $y = Sf \in V$ that satisfies

$$a(y,v) = (f,v)_{L^2(\Omega)} \qquad \forall v \in V := H_0^1(\Omega)$$

where $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is the bilinear form defined by

$$a(y,v) := \int_\Omega A\nabla y \cdot \nabla v + b \cdot \nabla y v + cyv. \tag{2.5}$$

The finite element solution $y_h = S_h f$ is given by the unique element of V_h that satisfies

$$a(y_h, v_h) = (f, v_h)_{L^2(\Omega)} \qquad \forall v_h \in V_h. \tag{2.6}$$

We recall the well-known result for the finite element error in the L^2-norm (see [3], [16], [17]).

Theorem 2.3. *Let y and y_h be the solution of* (2.5) *and* (2.6)*, respectively. On a mesh of type* (2.4) *with grading parameter $\mu < \lambda$ the estimate*

$$\|y - y_h\|_{L^2(\Omega)} + h\|y - y_h\|_{H^1(\Omega)} \leq ch^2|y|_{V^{2,2}_\beta(\Omega)} \leq ch^2\|f\|_{L^2(\Omega)} \tag{2.7}$$

is valid for $\beta > 1 - \lambda$.

The main result of this section is given in the following theorem.

Theorem 2.4. *Let y be the solution of the boundary value problem* (2.1) *with a right-hand side $f \in C^{0,\sigma}(\bar{\Omega})$. The finite element error can be estimated by*

$$\|y - y_h\|_{L^\infty(\Omega)} \leq ch^2|\ln h|\|f\|_{C^{0,\sigma}(\bar{\Omega})} \tag{2.8}$$

on finite element meshes with grading parameter $\mu < \lambda/2$.

Remark 2.5. *In order to achieve the L^∞-error estimate* (2.8) *a stronger mesh grading is necessary than in case of the L^2-error (comp. Theorem 2.3). A mesh is graded if $\mu < 1$. This means that the condition $\mu < \frac{\lambda}{2}$ yields a graded mesh not only in the case of a reentrant corner but also for corners with interior angle $\omega \geq \omega_0$.*

The remainder of this section concerns the proof of Theorem 2.4. During the error analysis we split Ω in different subsets. Therefore we write

$$\Omega = \bigcup_{j=0}^{I} \Omega_j$$

where $\Omega_I = \{x : |x| \leq d_I\}$, $\Omega_j = \{x : d_{j+1} \leq |x| \leq d_j\}$ for $j = 1, \ldots, I - 1$ and $\Omega_0 = \Omega \backslash \cup_{j=1}^{I} \Omega_j$ (see also Fig. 1). We set the radii d_j to $d_j = 2^{-j}$ $(j = 1, \ldots, I)$. The largest index I is chosen such that

$$d_I = c_I h^{1/\mu}, \tag{2.9}$$

what means $I \sim \log \frac{1}{h}$. For an appropriate choice of c_I we refer to Lemma 2.7. Further, we introduce the extended domains

$$\Omega'_j = \Omega_{j-1} \cup \Omega_j \cup \Omega_{j+1}.$$

with the obvious modification for $j = 0$ and $j = I$ and the subdomain meshsizes

$$h_j = \max_{T \in \Omega_j} h_T.$$

Lemma 2.6. *For the mesh introduced in* (2.4) *one has in Ω_j a family of quasi-uniform meshes with local mesh parameter*

$$c_1 h^{1/\mu} \leq h_I \leq c_2 c_I^{1-\mu} h^{1/\mu} \tag{2.10}$$

$$2^{\mu-1} c_1 h d_j^{1-\mu} \leq h_j \leq c_2 h d_j^{1-\mu} \quad j = 0 \ldots I - 1. \tag{2.11}$$

with constants c_1 and c_2 from (2.4) *and c_I from* (2.9).

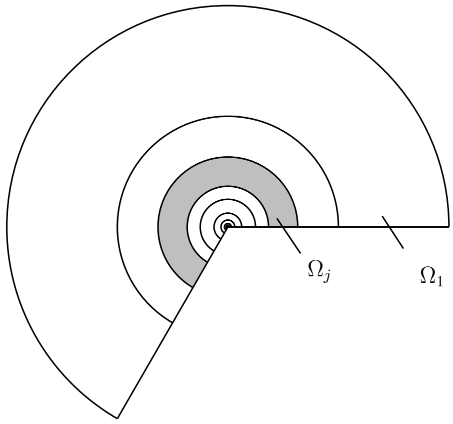

Figure 1: Ω is splitted in subsets Ω_j

Proof. For an element $T \subset \Omega_I$ one has $0 \le r_T \le d_I$ and therefore according to (2.4) the inequality $c_1 h^{1/\mu} \le h_T \le c_2 h d_I^{1-\mu}$ is valid. With (2.9) this yields (2.10). For $T \subset \Omega_j$ ($j \ne I$) one has $d_{j+1} \le r_T \le d_j$ and therefore $c_1 h d_{j+1}^{1-\mu} \le h_T \le c_2 h d_j^{1-\mu}$. Since $d_{j+1} = \frac{1}{2} d_j$ assertion (2.11) follows. □

Lemma 2.7. *For every fixed $c_0 < 1$ the constant c_I in (2.9) can be chosen, such that*

$$h_j d_j^{-1} \le c_0$$

for every $j = 0, \ldots, I$.

Proof. It follows for $c_I \ge \left(\frac{c_2}{c_0}\right)^{1/\mu}$

$$h_I d_I^{-1} \le \frac{c_2 c_I^{1-\mu} h^{1/\mu}}{c_I h^{1/\mu}} = c_2 c_I^{-\mu} \le c_0,$$

and for $j = 0, \ldots, I-1$

$$h_j d_j^{-1} \le c_2 h d_j^{1-\mu} d_j^{-1} = c_2 h d_j^{-\mu} \le c_2 h d_I^{-\mu} = c_2 h (c_I h^{1/\mu})^{-\mu} = c_2 c_I^{-\mu} \le c_0,$$

what proves the assertion. □

In the following we always have to distinguish between the domains near the corner and the domains away from the corner. In Lemma 2.10 we prove an estimate for the local error $\|y - y_h\|_{L^\infty(\Omega_J)}$ subject to the L^2- and H^1-error in the extended domain Ω'_J. The results in Lemma 2.12 show, that this H^1-error can be estimated with respect to the L^2-error in Ω'_J. Finally Lemma 2.13 gives an upper bound for $\|y - y_h\|_{L^2(\Omega'_J)}$, so that we can complete the proof of Theorem 2.4. Before we continue our considerations with an auxiliary result, we recall a lemma from Il'in [7].

Lemma 2.8. *Let G be a domain and ∂G satisfy a cone condition with radius $d > 0$. Then there exists a constant c, such that*

$$\|g\|_{C(\bar{G})} \leq c \cdot \|g\|_{H^1(G)} \cdot \begin{cases} 1 + |\ln q|^{1/2} & \text{for } q \leq d \\ 1 & \text{for } q > d \end{cases}$$

holds for all $g \in H^1(G)$ with $\nabla g \in L^\infty(G)$, where

$$q = \|g\|_{H^1(G)} \|\nabla g\|_{L^\infty(G)}^{-1}.$$

Lemma 2.9. *For every $v_h \in V_h$ and every $J \in \{1, \dots, I\}$ the estimates*

$$\|v_h\|_{L^\infty(\Omega_J)} \leq c|\ln h_J|^{1/2} \|v_h\|_{H^1(\Omega'_J)} \quad \forall J \in \{1, \dots, I\} \tag{2.12}$$

$$\|v_h\|_{L^\infty(\Omega)} \leq c|\ln h|^{1/2} \|v_h\|_{H^1(\Omega)} \tag{2.13}$$

are valid.

Proof. For quasi-uniform meshes a similar proof can be found in [24]. For an element T with $\bar{T} \cap \Omega_J \neq \emptyset$ one has

$$\|\nabla v_h\|_{L^\infty(T)} \leq c h_J^{-1} \|\nabla v_h\|_{L^2(T)}$$

and therefore

$$\|v_h\|_{H^1(T)} \|\nabla v_h\|_{L^\infty(T)}^{-1} \geq c^{-1} h_J \|v_h\|_{H^1(T)} \|\nabla v_h\|_{L^2(T)}^{-1} \geq c^{-1} h_J.$$

Now we can apply Lemma 2.8 and get

$$\|v_h\|_{L^\infty(T)} \leq c|\ln h_J|^{1/2} \|v_h\|_{H^1(T)}.$$

Assume that v_h admits its maximum in Ω_J in the element $\bar{T}_* \subset \Omega'_J$. Then one can estimate

$$\|v_h\|_{L^\infty(\Omega_J)} = \|v_h\|_{L^\infty(T_*)} \leq c|\ln h_J|^{1/2} \|v_h\|_{H^1(T_*)} \leq c|\ln h_J|^{1/2} \|v_h\|_{H^1(\Omega'_J)}$$

and the assertion (2.12) is shown. If one assumes now that v_h admits its maximum in Ω in the element $\bar{T}_* \subset \bar{\Omega}$, inequality (2.13) follows similarly,

$$\|v_h\|_{L^\infty(\Omega)} = \|v_h\|_{L^\infty(T_*)} \leq c|\ln h_T|^{1/2} \|v_h\|_{H^1(T)} \leq c|\ln h|^{1/2} \|v_h\|_{H^1(\Omega)}.$$

In the last step we have used $h_T \geq h^{1/\mu}$ and therefore $|\ln h_T| \leq c|\ln h|$. □

Lemma 2.10. *For $y \in V_\beta^{2,2}(\Omega_J) \cap V_\gamma^{2,\infty}(\Omega_J)$ with $\beta = 1 - \lambda + \delta$, $\gamma = 2 - \lambda + \delta$, $\mu = \frac{\lambda}{2} - \delta'$ and $\delta < 2\delta'$ the estimates*

$$\|y - y_h\|_{L^\infty(\Omega_J)} \leq c \left(h^2 |\ln h| |y|_{V_\gamma^{2,\infty}(\Omega''_J)} + d_J^{-1} \|y - y_h\|_{L^2(\Omega'_J)} \right) \quad \text{for } J = 0, 1, \dots, I-3, \tag{2.14}$$

$$\|y - y_h\|_{L^\infty(\Omega_J)} \leq c \left(|\ln h|^{1/2} h^2 \|y\|_{V_\gamma^{2,\infty}(\Omega'_J)} + |\ln h|^{1/2} \|y - y_h\|_{H^1(\Omega'_J)} \right) \quad \text{for } J \geq I-2 \tag{2.15}$$

are valid.

Proof. Let us first consider the case $J < I-2$, where one is away from the corner. We use the estimate

$$\|y-y_h\|_{L^\infty(\Omega_J)} \leq c\left(|\ln h|\min_{\chi\in V_h}\|y-\chi\|_{L^\infty(\Omega'_J)} + d_J^{-1}\|y-y_h\|_{L^2(\Omega'_J)}\right). \tag{2.16}$$

This result follows from Theorem 5.1 in [19] as shown in the proof of Corollary 5.1 of that paper, where the authors have already inserted an interpolation error estimate. If one chooses $l=0$, $N=2$, $p=0$ and $q=2$ in that corollary inequality (2.16) follows from writing $y-y_h$ as $y-\chi-y_h+\chi$. If one assumes that $y_h - I_h y$ admits its maximum in Ω'_J in $x_0 \in \bar{T}_* \subset \Omega''_J$, one can conclude

$$\begin{aligned}\|y-I_hy\|_{L^\infty(\Omega'_J)} &= \|y-I_hy\|_{L^\infty(T_*)}\\ &\leq ch_{T_*}^2|y|_{W^{2,\infty}(T_*)}\\ &\leq ch_J^2|y|_{W^{2,\infty}(\Omega''_J)}\\ &\sim h^2 d_J^{2-2\mu}|y|_{W^{2,\infty}(\Omega''_J)}\\ &\sim h^2 d_J^{2-2\mu-\gamma}|y|_{V_\gamma^{2,\infty}(\Omega''_J)}.\end{aligned}$$

Since $2-2\mu-\gamma = \lambda-\delta-2(\frac{\lambda}{2}-\delta') = 2\delta'-\delta > 0$ this yields

$$\|y-I_hy\|_{L^\infty(\Omega'_J)} \leq ch^2|y|_{V_\gamma^{2,\infty}(\Omega''_J)}.$$

Then the assertion (2.14) follows from inequality (2.16).

Let us now consider the case of $J = I, I-1, I-2$. With the triangle inequality it follows

$$\|y-y_h\|_{L^\infty(\Omega_J)} \leq \|y\|_{L^\infty(\Omega_J)} + \|y_h\|_{L^\infty(\Omega_J)}. \tag{2.17}$$

We estimate the two terms separately. Assume that y admits its maximum in Ω_J in $x_0 \in \Omega_J$ and that $x_0 \in \bar{T}_*$ with $\bar{T}_* \subset \Omega'_J$. Then one can conclude from the embedding (2.3)

$$\|y\|_{L^\infty(\Omega_J)} \leq \|y\|_{L^\infty(T_*)} = \|\hat{y}\|_{L^\infty(\hat{T})} \leq c\|\hat{y}\|_{V_\gamma^{2,\infty}(\hat{T})} \sim h^{\frac{2-\gamma}{\mu}}\|y\|_{V_\gamma^{2,\infty}(T_*)} \leq ch^2\|y\|_{V_\gamma^{2,\infty}(\Omega'_J)}. \tag{2.18}$$

since $2-\gamma = \lambda-\delta = 2\mu+2\delta'-\delta > 2\mu$. In order to estimate the second term of (2.17) we use Lemma 2.9 and get the inequality

$$\|y_h\|_{L^\infty(\Omega_J)} \leq c|\ln h_J|^{1/2}\|y_h\|_{H^1(\Omega'_J)} \leq c|\ln h|^{1/2}\|y_h\|_{H^1(\Omega'_J)}. \tag{2.19}$$

where we have used $h_J \sim h^{1/\mu}$ in the last step. We use again the triangle inequality to estimate

$$\|y_h\|_{H^1(\Omega'_J)} \leq \|y\|_{H^1(\Omega'_J)} + \|y-y_h\|_{H^1(\Omega'_J)}. \tag{2.20}$$

In order to estimate the first part of the right-hand side of this inequality we continue with

$$\begin{aligned}
\|y\|_{H^1(\Omega'_J)} &\sim \|r^{-\beta} r^{\beta}\nabla y\|_{L^2(\Omega'_J)} + \|r^{1-\beta}r^{\beta-1}y\|_{L^2(\Omega'_J)} \\
&\le \|r^{-\beta}\|_{L^2(\Omega'_J)}\|r^{\beta}\nabla y\|_{L^\infty(\Omega'_J)} + \|r^{1-\beta}\|_{L^2(\Omega'_J)}\|r^{\beta-1}y\|_{L^\infty(\Omega'_J)} \\
&\le c h_I^{1-\beta}\|y\|_{V^{1,\infty}_{\beta}(\Omega'_J)} \\
&\le c h^{\frac{1-\beta}{\mu}}\|y\|_{V^{2,\infty}_{\beta+1}(\Omega'_J)} \\
&\le c h^2\|y\|_{V^{2,\infty}_{\gamma}(\Omega'_J)},
\end{aligned}$$

since $1-\beta = \lambda - \delta = 2\mu + 2\delta' - \delta > 2\mu$ and $\gamma = \beta - 1$. With this estimate one has from (2.19) and (2.20)

$$\|y_h\|_{L^\infty(\Omega_J)} \le c\left(|\ln h|^{1/2}h^2\|y\|_{V^{2,\infty}_{\gamma}(\Omega'_J)} + |\ln h|^{1/2}\|y-y_h\|_{H^1(\Omega'_J)}\right)$$

what yields together with (2.17) and (2.18) the desired result. □

Lemma 2.11. *The estimate*

$$\|y-y_h\|_{H^1(\Omega_J)} \le c\left(\|y-I_hy\|_{H^1(\Omega'_J)} + d_J^{-1}\|y-I_hy\|_{L^2(\Omega'_J)} + d_J^{-1}\|y-y_h\|_{L^2(\Omega'_J)}\right)$$

is valid for $J = 0, 1, \dots, I$.

Proof. The assertion follows from Lemma 7.2 of [20] by setting $D_1 = \Omega_J$, $D = \Omega'_J$ and $p = 0$ in that lemma. It follows from Lemma 2.6 and the explanations in Example 4 of Section 9 in [20], that the result is applicable with our finite element space. Notice, that the proof is only given for $L = -\Delta$. For an extension to general elliptic operators the proof has to be modified at two points, where the bilinear form explicitly steps in. After equation (7.7) in that proof one has to substitute the estimate of $\|v_h\|^2_{1,D_1}$ by

$$\begin{aligned}
&\|v_h\|^2_{1,D_1} \le \|\omega v_h\|^2_{1,D} \\
&\quad\le ca(v_h,\omega^2 v_h) + c\int_\Omega (\nabla\cdot(A\nabla w))\omega)\omega v_h + v_h\left(2A\nabla\omega\cdot\nabla(\omega v_h)\right) + \int_{D\setminus D_1} v_h^2\omega b\cdot\nabla\omega.
\end{aligned}$$

Notice, that in [20] A is the bilinear form while in our setting a is the bilinear form and A the coefficient matrix in the operator L. Since $A \in W^{1,\infty}(\Omega, \mathbb{R}^{2,2})$ and $\|b\cdot\nabla\omega\|_{L^\infty(\Omega)} \le C$ it follows

$$\|\omega v_h\|^2_{1,D_1} \le Ch\|v_h\|^2_{1,D} + C\|v_h\|_{0,D\setminus D_1}\|\omega v_h\|^2_{1,D_1}$$

and therefore equation (7.8) in [20]. The second point is after expression (7.10), where the equation for $(\omega v_h, \varphi)$ has to be substituted by

$$(\omega v_h,\varphi) = a(\omega v_h,\psi) = a(v_h,\omega\psi) + \int_D v_h\left(\psi\nabla\cdot(A\nabla\omega) + 2A\nabla\omega\cdot\nabla\psi + \psi b\cdot\nabla\omega\right).$$

Again it follows from the fact $A \in W^{1,\infty}(\Omega, \mathbb{R}^{2,2})$ and $\|b\cdot\nabla\omega\|_{L^\infty(\Omega)} \le C$ that the argumentation can be completed as for the Laplace operator. □

Lemma 2.12. *For $y \in V^{2,2}_\beta(\Omega_J) \cap V^{2,\infty}_\gamma(\Omega_J)$, $\beta = 1-\lambda+\delta$, $\gamma = 2-\lambda+\delta$ and $\mu = \frac{\lambda}{2} - \delta'$ ($\delta < 2\delta'$) the estimates*

$$\|y - y_h\|_{H^1(\Omega_J)} \le c\left(hd_J^{2-\gamma-\mu}|y|_{V^{2,\infty}_\gamma(\Omega''_J)} + d_J^{-1}\|y-y_h\|_{L^2(\Omega'_J)}\right) \quad \text{for } J = 0,1,\ldots,I-3 \tag{2.21}$$

$$\|y - y_h\|_{H^1(\Omega_J)} \le c\left(h^2\|y\|_{V^{2,\infty}_\gamma(\Omega''_J)} + d_J^{-1}\|y-y_h\|_{L^2(\Omega'_J)}\right) \quad \text{for } J = I, I-1, I-2 \tag{2.22}$$

are valid.

Proof. From Lemma 2.11 we have

$$\begin{aligned}\|y - y_h\|_{H^1(\Omega_J)} &\le c\left(\|y - I_hy\|_{H^1(\Omega'_J)} + d_J^{-1}\|y - I_hy\|_{L^2(\Omega'_J)} + d_J^{-1}\|y-y_h\|_{L^2(\Omega'_J)}\right)\\ &\le c\left(\|y - I_hy\|_{H^1(\Omega'_J)} + \|y - I_hy\|_{L^\infty(\Omega'_J)} + d_J^{-1}\|y-y_h\|_{L^2(\Omega'_J)}\right).\end{aligned} \tag{2.23}$$

where we have used $\|y - I_hy\|_{L^2(\Omega'_J)} \le c|\Omega'_J|^{1/2}\|y - I_hy\|_{L^\infty(\Omega'_J)}$ and $|\Omega'_J| \sim d_J^2$. In the case $J = 0,\ldots,I-3$ one has $y \in W^{1,\infty}(\Omega''_J)$. If one assumes that the maximum of $y - I_hy$ and its first derivatives in Ω'_J is admitted in $x_0 \in T_* \subset \Omega''_J$, one can conclude

$$\begin{aligned}\|y - I_hy\|_{H^1(\Omega'_J)} &\le c|\Omega'_J|^{1/2}\|y - I_hy\|_{W^{1,\infty}(\Omega'_J)}\\ &\le cd_J\|y - I_hy\|_{W^{1,\infty}(T_*)}\\ &\le cd_Jh_{T_*}|y|_{W^{2,\infty}(T_*)}\\ &\le cd_J^{1-\gamma}h_{T_*}|y|_{V^{2,\infty}_\gamma(T_*)}\\ &\le cd_J^{1-\gamma}h_J|y|_{V^{2,\infty}_\gamma(\Omega''_J)}.\end{aligned}$$

With $h_J = hd_J^{1-\mu}$ we arrive at

$$\|y - I_hy\|_{H^1(\Omega'_J)} \sim hd_J^{2-\gamma-\mu}|y|_{V^{2,\infty}_\gamma(\Omega''_J)}. \tag{2.24}$$

Like in the proof of Lemma 2.10 the second term on the right-hand side of inequality (2.23) can be estimated by

$$\begin{aligned}\|y - I_hy\|_{L^\infty(\Omega'_J)} &\sim h^2d_J^{2-2\mu-\gamma}|y|_{V^{2,\infty}_\gamma(\Omega''_J)}\\ &\le chd_J^{-\mu}\cdot hd_J^{2-\gamma-\mu}|y|_{V^{2,\infty}_\gamma(\Omega''_J)}\\ &\le chd_J^{2-\gamma-\mu}|y|_{V^{2,\infty}_\gamma(\Omega''_J)}\end{aligned}$$

since $hd_J^{-\mu} = h_Jd_J^{-1} \le c_0$ by Lemma 2.7. This last estimate yields together with the estimate (2.24) and the inequality (2.23) the assertion (2.21).

In the case of $J = I, I-1, I-2$ one can write

$$\|y - I_hy\|_{H^1(\Omega_J)} \le \|y\|_{H^1(\Omega_J)} + \|I_hy\|_{H^1(\Omega_J)} \tag{2.25}$$

For the first term one has like in the proof of Lemma 2.10

$$\|y\|_{H^1(\Omega_J)} \le ch^2 \|y\|_{V^{2,\infty}_\gamma(\Omega_J)}.$$

For the second term we conclude with the inverse inequality, the estimate $\|I_h y\|_{L^\infty(\Omega_J)} \le \|y\|_{L^\infty(\Omega_J)}$ and the embedding (2.3)

$$\begin{aligned} \|I_h y\|_{H^1(\Omega_J)} &\le c h_J^{-1} \|I_h y\|_{L^2(\Omega_J)} \\ &\le c h_J^{-1} d_J \|y\|_{L^\infty(\Omega_J)} \le c\|y\|_{L^\infty(T_*)} \\ &\le c h_J^{2-\gamma} \|y\|_{V^{2,\infty}_\gamma(T_*)} \le ch^2 \|y\|_{V^{2,\infty}_\gamma(\Omega'_J)} \end{aligned} \tag{2.26}$$

since $d_J \sim h_J$, $h_J \sim h^{1/\mu}$ and $2-\gamma = \lambda - \delta = 2\mu + 2\delta' - \delta > 2\mu$. The inequalities (2.25)–(2.26) yield

$$\|y - I_h y\|_{H^1(\Omega_J)} \le ch^2 \|y\|_{V^{2,\infty}_\gamma(\Omega'_J)}. \tag{2.27}$$

In order to estimate the second term in (2.23) in that case, we conclude similarly to (2.26)

$$\|y - I_h y\|_{L^\infty(\Omega'_J)} \le \|y - I_h y\|_{L^\infty(T_*)} \le c\|y\|_{L^\infty(T_*)} \le ch^2 \|y\|_{V^{2,\infty}_\gamma(\Omega''_J)}.$$

This estimate yields together with (2.23) and (2.27) the desired inequality (2.22). □

Lemma 2.13. *Under the conditions of Lemma 2.12 the inequality*

$$d_J^{-1} \|y - y_h\|_{L^2(\Omega'_J)} \le ch^2 |\log h|^{1/2} \|y\|_{V^{2,\infty}_\gamma(\Omega)}. \tag{2.28}$$

is valid for $J = 0, \dots, I$.

Proof. For this proof we introduce the abbreviation $e := y - y_h$. One has the equality

$$\|e\|_{L^2(\Omega'_J)} = \sup_{\substack{\varphi \in C_0^\infty(\Omega'_J) \\ \|\varphi\|_{L^2(\Omega'_J)} = 1}} (e, \varphi). \tag{2.29}$$

For every such function φ we consider the boundary value problem

$$\begin{aligned} -\Delta v &= d_J^{-1} \varphi \qquad \text{in } \Omega'_J \\ v &= 0 \qquad \text{on } \partial\Omega'_J \end{aligned}$$

Then one can conclude

$$d_J^{-1}(e, \varphi) = (e, d_J^{-1}\varphi) = (\nabla e, \nabla v) = (\nabla e, \nabla(v - I_h v)) \le \|e\|_{H^1(\Omega'_J)} |v - I_h v|_{H^1(\Omega'_J)}. \tag{2.30}$$

We introduce the domains

$$\Omega_{J,h} = \{T : T \cap \Omega_J \ne \emptyset\}.$$

Now we distinguish the two cases $J \leq I-3$ and $J = I, I-1, I-2$. We begin with $J \leq I-3$. It follows from the standard interpolation theory

$$|v - I_h v|^2_{H^1(\Omega_J)} \leq \sum_{T\in\Omega_{J,h}} |v - I_h v|^2_{H^1(T)} \leq c \sum_{T\in\Omega_{J,h}} h_J^2 |v|^2_{H^2(T)} \leq h_J^2 |v|^2_{H^2(\Omega'_J)} \tag{2.31}$$

In order to use an a priori estimate where the constant does not depend on J we introduce $\hat{\Omega} := \{(r,\rho), 1/2 < r < 1, 0 < \rho < \omega\}$. The coordinate transformation $x \to d_J \hat{x}$ yields the boundary value problem

$$-\frac{1}{d_J^2}\Delta\hat{v} = d_J^{-1}\hat{\varphi} \quad \text{in } \hat{\Omega}, \qquad \hat{v} = 0 \quad \text{in } \partial\hat{\Omega}$$

This results in the a priori estimate

$$|v|_{H^2(\Omega'_J)} = \frac{1}{d_J^2}\frac{|\Omega_J|}{|\hat{\Omega}|}|\hat{v}|_{H^2(\hat{\Omega})} \leq \frac{c}{d_J^2}\frac{|\Omega_J|}{|\hat{\Omega}|}\|d_J\hat{\varphi}\|_{L^2(\hat{\Omega})} \leq \frac{c}{d_J}\|\varphi\|_{L^2(\Omega'_J)} = \frac{c}{d_J}$$

with c independent of J. With (2.31) one gets

$$|v - I_h v|_{H^1(\Omega_J)} \leq c h_J d_J^{-1}.$$

Since $h_J \sim h'_J$ and $d_J \sim d'_J$ it follows

$$|v - I_h v|_{H^1(\Omega'_J)} \leq c h_J d_J^{-1}. \tag{2.32}$$

With this inequality one can estimate together with (2.29) and (2.30)

$$\|d_J^{-1} e\|_{L^2(\Omega'_J)} \leq c h_J d_J^{-1} \|e\|_{H^1(\Omega'_J)}.$$

If we apply Lemma 2.12 to this inequality and take Lemma 2.6 into account, we can continue

$$\begin{aligned}
\|d_J^{-1} e\|_{L^2(\Omega'_J)} &\leq c h_J d_J^{-1}\left(h d_J^{2-\gamma-\mu}|u|_{V_\gamma^{2,\infty}(\Omega'''_J)} + \|d_J^{-1}e\|_{L^2(\Omega''_J)}\right)\\
&\leq c\left(h^2 d_J^{2-\gamma-2\mu}|u|_{V_\gamma^{2,\infty}(\Omega'''_J)} + h_J d_J^{-1}\|d_J^{-1}e\|_{L^2(\Omega''_J)}\right)\\
&\leq c\left(h^2 |u|_{V_\gamma^{2,\infty}(\Omega'''_J)} + h_J d_J^{-1}\|d_J^{-1}e\|_{L^2(\Omega''_J)}\right).
\end{aligned}$$

In the last estimate we have used $2 - \gamma - 2\mu = \lambda - \delta - 2\left(\frac{\lambda}{2} - \delta'\right) = 2\delta' - \delta > 0$. With Lemma 2.7 we can conclude for an arbitrary, but fixed $c_0 < 1$ the inequality

$$\|d_J^{-1} e\|_{L^2(\Omega'_J)} \leq c h^2 |u|_{V_\gamma^{2,\infty}(\Omega'''_J)} + c_0\|d_J^{-1}e\|_{L^2(\Omega''_J)} \qquad (J = 0, 1, \ldots, I-2). \tag{2.33}$$

Let us now consider the case of $J = I, I-1$. For $T \in \Omega_{J,h}$ one gets with $\beta = \gamma - 1$ from [2, Proof of Theorem 3.2] the estimate

$$|v - I_h v|_{H^1(T)} \leq c h_J^{1-\beta}|v|_{V_\beta^{2,2}(T)}.$$

Summing up over all elements yields with $\Omega_{J,h} = \cup_{T\cap\Omega_J\neq\emptyset}\bar{T}$

$$|v - I_h v|_{H^1(\Omega_J)} \le |v - I_h v|_{H^1(\Omega_{J,h})} \le c h_J^{1-\beta} |v|_{V^{2,2}_\beta(\Omega_{J,h})} \le c h_J^{1-\beta} |v|_{V^{2,2}_\beta(\Omega'_J)}. \tag{2.34}$$

With the same transformation as above and an a priori estimate for boundary value problems in domains with conical points (see [9]) it follows

$$|v|_{V^{2,2}_\beta(\Omega'_J)} \sim d_J^{\beta-2} \frac{|\Omega_J|}{|\hat{\Omega}|} \|\hat{v}\|_{V^{2,2}_\beta(\hat{\Omega})} \le c d_J^{\beta-2} \frac{|\Omega_J|}{|\hat{\Omega}|} \|d_J \hat{\varphi}\|_{L^2(\hat{\Omega})} \le c d_J^{\beta-1} \|\varphi\|_{L^2(\Omega'_J)} \sim d_J^{\beta-1}.$$

This yields together with (2.34) and the same argumentation as above

$$|v - I_h v|_{H^1(\Omega'_J)} \le c(h_J d_J^{-1})^{1-\beta}$$

and (2.29) and (2.30) result in

$$\|d_J^{-1} e\|_{L^2(\Omega'_J)} \le c(h_J d_J^{-1})^{1-\beta} \|e\|_{H^1(\Omega'_J)}.$$

Now we apply Lemma 2.12 to this inequality and arrive at

$$\|d_J^{-1} e\|_{L^2(\Omega'_J)} \le c\left((h_J d_J^{-1})^{1-\beta} h^2 \|y\|_{V^{2,\infty}_\gamma(\Omega'''_J)} + (h_J d_J^{-1})^{1-\beta} \|d_J^{-1} e\|_{L^2(\Omega''_J)} \right)$$

According to Lemma 2.7 it follows for h small enough $h_J d_J^{-1} < \frac{1}{c} c_0^{1/(1-\beta)}$ and therefore

$$\|d_J^{-1} e\|_{L^2(\Omega'_J)} \le c h^2 \|y\|_{V^{2,\infty}_\gamma(\Omega'''_J)} + c_0 \|d_J^{-1} e\|_{L^2(\Omega''_J)} \tag{2.35}$$

Summing up over all Ω_J, $J = 0, 1, \dots, I$ one has with (2.33) and (2.35)

$$\|d_J^{-1} e\|^2_{L^2(\Omega)} \le c h^4 \|y\|^2_{V^{2,\infty}_\gamma(\Omega)} \cdot \sum_{i=0}^{I} 1 + c_1 \|d_J^{-1} e\|^2_{L^2(\Omega)}.$$

If one has chosen c_0 small enough also $c_1 < 1$ is valid. Since $I \sim |\ln h|$ the inequality

$$\|d_J^{-1} e\|^2_{L^2(\Omega)} \le c h^4 |\ln h| \|y\|^2_{V^{2,\infty}_\gamma(\Omega)}$$

follows, what results in the desired estimate (2.28). □

Remark 2.14. *If one denotes by $h_{J'}$ the element size in Ω'_J and by $h_{J''}$ the element size in $\Omega''_J = (\Omega'_J)'$ one has*

$$h_J \sim \frac{1}{2} h_{J'} \sim \frac{1}{4} h_{J''}.$$

Therefore in Lemmata 2.12 and 2.13 one can substitute Ω_J by Ω'_J and Ω'_J by Ω''_J.

Now we are able to prove Theorem 2.4.

Proof. Let us first consider the error in $\Omega_I \cup \Omega_{I-1}$. From Lemma 2.10 we have together with Lemmata 2.12 and 2.13 and Remark 2.14 for $J = I, I-1, I-2$ the estimate

$$\|y - y_h\|_{L^\infty(\Omega_J)} \le c h^2 |\ln h| \|y\|_{V^{2,\infty}_\gamma(\Omega)}. \tag{2.36}$$

For Ω_J, $J \neq I, I-1, I-2$ we conclude from Lemma 2.10, 2.13 and Remark 2.14 the same estimate. The assertion follows with the a priori estimate of Theorem 2.2. □

3 Error estimates for the optimal control problem

3.1 General considerations

With the estimates for the state equation from Section 2, we are now able to derive L^∞-error estimates for the optimal control problem (1.1) – (1.6). This problem admits a unique solution $\bar{u}$, that satisfies the variational inequality

$$(\bar{p} + \nu\bar{u}, u - \bar{u})_{L^2(\Omega)} \geq 0 \qquad \forall u \in U_{\text{ad}}$$

Here, $\bar{p} = P\bar{u}$ denotes the corresponding adjoint state. This inequality is equivalent to the expression

$$\bar{u} = \Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}\bar{p}\right) \tag{3.1}$$

where the projection $\Pi_{[u_a,u_b]}$ is given by

$$\Pi_{[u_a,u_b]}f(x) := \max(u_a, \min(u_b, f(x))).$$

Remark 3.1. *From [1, Remark 2] one has for $p < 2/(2-\lambda)$ that y is contained in the classical Sobolev space $W^{2,p}(\Omega)$. For a number $\sigma \in (0,1]$ small enough, this space, for $p > 1$, is embedded in $C^{0,\sigma}(\Omega)$. Therefore we can conclude for the solution of* (2.1)

$$y \in C^{0,\sigma}(\bar{\Omega}) \qquad \textit{if } f \in L^p(\Omega),\ p > 1.$$

Notice that this assertion also holds for $p > 2/(2-\lambda)$ because the data can, of course, be smoother than necessary.

With an argumention analogous to the one after Remark 2 in [1] one can show, that for the optimal control problem (1.1)–(1.6) *the estimates*

$$\|Su\|_{C^{0,\sigma}(\bar{\Omega})} \leq c\|u\|_{L^\infty(\Omega)} \leq c\|u\|_{C^{0,\sigma}(\bar{\Omega})},$$
$$\|Pu\|_{C^{0,\sigma}(\bar{\Omega})} \leq c\|Su + y_d\|_{L^\infty(\Omega)} \leq c\left(\|u\|_{C^{0,\sigma}(\bar{\Omega})} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right)$$

are valid.

Remark 3.2. *From $\bar{p} = P\bar{u} \in C^{0,\sigma}(\bar{\Omega})$ and* (3.1) *we obtain that the optimal control $\bar{u}$ belongs to $C^{0,\sigma}(\bar{\Omega})$, $\sigma \in (0,1]$.*

In the following, we will investigate two different types of discretization of the optimal control problem (1.1)–(1.6), namely the approach of variational discretization suggested by Hinze in [6] and the fully discrete approach originally introduced by Meyer and Rösch in [12].

We denote by S_h and S_h^* the finite element solution operator corresponding to S and S^* respectively. In the case of $\mu > \frac{1}{2}$ the following lemma about the boundedness of S_h and S_h^* was already proven in [1]. Since in our setting here $\mu < \lambda/2$ may be required (comp. Section 1), we cannot fulfill this condition for $\lambda \in [1/2, 1]$. Therefore we give in the following a more involved proof of [1, Lemma 3] without the condition $\mu > \frac{1}{2}$.

Lemma 3.3. *Let T_h be a graded mesh with parameter $\mu < \lambda$. The norms of the discrete solution operators S_h and S_h^* are bounded,*

$$
\begin{aligned}
\|S_h\|_{L^2(\Omega)\to L^\infty(\Omega)} &\le c, & \|S_h^*\|_{L^2(\Omega)\to L^\infty(\Omega)} &\le c,\\
\|S_h\|_{L^2(\Omega)\to L^2(\Omega)} &\le c, & \|S_h^*\|_{L^2(\Omega)\to L^2(\Omega)} &\le c,\\
\|S_h\|_{L^2(\Omega)\to H_0^1(\Omega)} &\le c, & \|S_h^*\|_{L^2(\Omega)\to H_0^1(\Omega)} &\le c,\\
\|S_h\|_{L^\infty(\Omega)\to L^\infty(\Omega)} &\le c, & \|S_h^*\|_{L^\infty(\Omega)\to L^\infty(\Omega)} &\le c,
\end{aligned}
$$

where c is independent of h.

The remainder of this subsection is devoted to the proof of this lemma. For that, we use estimates of norms of a regularized Green function. We introduce the regularized Dirac function

$$
\delta^h := \begin{cases} |T_*|^{-1}\mathrm{sgn}(e) & \text{in } T_*,\\ 0 & \text{elsewhere,} \end{cases} \tag{3.2}
$$

where we abbreviated the finite element error by e,

$$
e := y - y_h.
$$

The regularized Green function g^h is defined as solution of

$$
a(\varphi, g^h) = (\delta^h, \varphi) \quad \forall \varphi \in V, \tag{3.3}
$$

and its discrete counterpart g_h^h by

$$
a(\varphi_h, g_h^h) = (\delta^h, \varphi_h) \quad \forall \varphi_h \in V_h. \tag{3.4}
$$

Lemma 3.4. *The norms of the regularized Green function can be estimated by*

$$
\|g^h\|_{L^\infty(\Omega)} \le c|\ln h| \tag{3.5}
$$

$$
\|g^h\|_{H^1(\Omega)} \le c|\ln h|^{1/2} \tag{3.6}
$$

$$
\|g^h\|_{V_\beta^{2,2}(\Omega)} \le ch^{-1} \tag{3.7}
$$

where $\beta := 1 - \mu > 1 - \lambda$ is the weight corresponding to the regularity in $V_\beta^{2,2}(\Omega)$, and the grading parameter satisfies $\mu < \lambda$.

Proof. Let $g(x)$ be the Green function with respect to an arbitrary point $x_+ \in \Omega$,

$$
a(\varphi, g) = \varphi(x_+) \quad \forall \varphi \in V. \tag{3.8}
$$

The Green function satisfies the following inequality

$$
|g(x)| \le c\left(\big|\ln|x - x_+|\big| + 1\right) \quad \forall x \in \Omega. \tag{3.9}
$$

In [4] it is proven that this estimate is valid on Lipschitz domains and that the constant c is independent of x_+. Using (3.8), (3.3), and (3.2) we get

$$|g^h(x_+)| = |a(g^h, g)| = |(\delta^h, g)| \le |T_*|^{-1} \int_{T_*} |g| \,\mathrm{d}x$$

In the case $\operatorname{dist}(x_+, T_*) > h_{T_*}$ we have $|x - x_+| \ge h_{T_*}$ and estimate (3.5) is obtained via

$$|T_*|^{-1} \int_{T_*} |g| \,\mathrm{d}x \le \max_{x\in T_*} |g(x)| \le c \max_{x\in T_*} \big(\big| \ln |x - x_+| \big| + 1 \big) \le c |\ln h_{T_*}| \le c |\ln h|,$$

since $h_{T_*} \ge c h^{1/\mu}$. In the case $\operatorname{dist}(x_+, T_*) \le h_{T_*}$ we calculate the integral by using polar coordinates centered in x_+,

$$|T_*|^{-1} \int_{T_*} |g| \,\mathrm{d}x \le c |T_*|^{-1} \int_0^{2h_{T_*}} (-\ln r) r \mathrm{d}r = c |T_*|^{-1} h_{T_*}^2 (c - \ln h_{T_*}) \le c |\ln h|$$

as above.

For the proof of (3.6) we use the coercivity of the bilinear form and the definitions (3.3) of g^h and (3.2) of δ_h,

$$c \|g^h\|_{H^1(\Omega)}^2 \le a(g^h, g^h) = (\delta^h, g^h) \le \|g^h\|_{L^\infty(\Omega)} \|\delta_h\|_{L^1(\Omega)} \le \|g^h\|_{L^\infty(\Omega)}.$$

With (3.5) we conclude (3.6).

The a priori estimate for the solution of the elliptic partial differential equation, and the definition (3.2) of δ_h give

$$\|g^h\|_{V_\beta^{2,2}(\Omega)} \le c \|r^\beta \delta^h\|_{L^2(\Omega)} \le c |T_*|^{-1} \|r^\beta\|_{L^2(T_*)}.$$

With $r \le d_J$, we can continue by

$$|T_*|^{-1} \|r^\beta\|_{L^2(T_*)} \le c |T_*|^{-1/2} d_J^\beta = c h^{-1}$$

since $|T_*|^{1/2} = c h_{T_*} = c h d_J^{1-\mu} = c h d_J^\beta$. In the other case, $J = I$, we calculate the L^2-norm and obtain

$$|T_*|^{-1} \|r^\beta\|_{L^2(T_*)} \le c |T_*|^{-1} h_{T_*}^{\beta+1} \le c h_{T_*}^{\beta-1} = c h^{-1}$$

since $h_{T_*} = c h^{1/\mu} = c h^{1/(1-\beta)}$. Thus (3.7) is proved. □

Corollary 3.5. *On meshes with grading parameter $\mu = 1 - \beta < \lambda$ the error estimates*

$$\|g^h - g_h^h\|_{H^1(\Omega)} \le c \tag{3.10}$$

$$\|g^h - g_h^h\|_{L^2(\Omega)} \le ch \tag{3.11}$$

hold.

Proof. Since the meshes are optimally graded, one has from Lemma 2.3

$$\begin{aligned}\|g^h - g_h^h\|_{H^1(\Omega)} &\leq ch\|r^\beta \nabla^2 g^h\|_{L^2(\Omega)}, \\ \|g^h - g_h^h\|_{L^2(\Omega)} &\leq ch^2\|r^\beta \nabla^2 g^h\|_{L^2(\Omega)},\end{aligned}$$

With (3.7) we get the assertion. □

Now we are able to prove Lemma 3.3.

Proof. First we prove the boundedness of $\|S_h\|_{L^2(\Omega)\to L^\infty(\Omega)}$. To this end, we consider a function $f \in L^2(\Omega)$ and an arbitrary but fixed finite element T_*. Then the three inequalities

$$\begin{aligned}\|y_h\|_{L^\infty(T_*)} &\leq c|T_*|^{-1}\|y_h\|_{L^1(T_*)}, \\ \|y_h\|_{L^1(T_*)} &\leq \|y - y_h\|_{L^1(T_*)} + \|y\|_{L^1(T_*)}, \\ \|y\|_{L^1(T_*)} &\leq |T_*|\|y\|_{L^\infty(T_*)}\end{aligned}$$

yield the estimate

$$\|y_h\|_{L^\infty(T_*)} \leq c|T_*|^{-1}\|y - y_h\|_{L^1(T_*)} + c\|y\|_{L^\infty(T_*)}. \tag{3.12}$$

By the definition of δ^h and g^h we get for the first term on the right-hand side of this inequality the equation

$$|T_*|^{-1}\|e\|_{L^1(T_*)} = (\delta_h, e) = a(e, g^h).$$

Using the Galerkin orthogonality and the fact that $e - I_h e = y - I_h y$ yields

$$|T_*|^{-1}\|e\|_{L^1(T_*)} = a(e, g^h - g_h^h) = a(e - I_h e, g^h - g_h^h) = a(y - I_h y, g^h - g_h^h)$$

With the Cauchy-Schwartz inequality we can continue

$$|T_*|^{-1}\|e\|_{L^1(T_*)} \leq c\|y - I_h y\|_{H^1(\Omega)}\|g^h - g_h^h\|_{H^1(\Omega)} \tag{3.13}$$

From finite element theory one knows that

$$\|y - I_h y\|_{H^1(\Omega)} \leq ch^\kappa\|f\|_{L^2(\Omega)} \tag{3.14}$$

with $\kappa = \min\{\frac{\lambda}{\mu}, 1\}$. Consequently, one can conclude from (3.13) together with (3.14) and Corollary 3.5

$$|T_*|^{-1}\|e\|_{L^1(T_*)} \leq ch.$$

This shows together with (3.12)

$$\|S_h f\|_{L^\infty(T_*)} \leq c\|f\|_{L^2(\Omega)}.$$

The boundedness of $\|S_h\|_{L^2(\Omega)\to L^2(\Omega)}$ and $\|S_h\|_{L^\infty(\Omega)\to L^\infty(\Omega)}$ follows then by the embedding theorem $L^\infty(\Omega) \hookrightarrow L^2(\Omega)$. The boundedness of $\|S_h\|_{L^2(\Omega)\to H_0^1(\Omega)}$ comes from the theory of weak solutions. The estimates for S_h^* follow by analogy. □

3.2 Approach of variational discretization

We consider as discretization the optimal control problem

$$J_h^s(\bar u_h^s) = \min_{u\in U_{\mathrm{ad}}} F(S_h u, u) \tag{3.15}$$
$$J_h^s(u) := \frac{1}{2}\|S_h u - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)}.$$

Remark 3.6. *The optimal control problem* (3.15) *admits a unique solution $\bar u_h^s$. In the following, we use the notation $\bar y_h^s = S_h\bar u_h^s$ and $\bar p_h^s = P_h\bar u_h^s$ for the optimal discrete state and adjoint state. The corresponding necessary and sufficient optimality condition is given by*

$$(\nu\bar u_h^s + \bar p_h^s, v - \bar u_h^s)_{L^2(\Omega)} \ge 0 \qquad \forall v \in U_{\mathrm{ad}}. \tag{3.16}$$

In the case of convex domains and quasi-uniform triangulations, it is shown in [6] that a finite element discretization of S with piecewise linear and globally continuous functions yields an approximation rate h^2 in the L^2-norm. This results extends to non-convex domains with graded meshes (see [1, Remark 5]). Therefore the following theorem is valid.

Theorem 3.7. *Let $\bar u$ and $\bar u_h^s$ be the solutions of* (1.1) *and* (3.15)*, respectively. If S is discretized on a mesh, that is graded according to* (2.4) *with $\mu < \lambda$, the estimate*

$$\|\bar u - \bar u_h^s\|_{L^2(\Omega)} \le ch^2\left(\|\bar u\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)}\right)$$

is valid.

With the L^2-error estimate at hand, one is now able to prove an L^∞-error estimate.

Theorem 3.8. *Let $\bar u_h^s$ be the discrete control introduced in* (3.15) *and $\bar y_h^s = S_h\bar u_h^s$ and $\bar p_h^s = P_h\bar u_h^s$ the associated state and adjoint state, respectively. On a family of meshes with grading parameter $\mu < \frac{\lambda}{2}$ the estimates*

$$\|\bar u - \bar u_h^s\|_{L^\infty(\Omega)} \le ch^2|\ln h|\left(\|\bar u\|_{C^{0,\sigma}(\bar\Omega)} + \|y_d\|_{C^{0,\sigma}(\bar\Omega)}\right) \tag{3.17}$$
$$\|\bar y - \bar y_h^s\|_{L^\infty(\Omega)} \le ch^2|\ln h|\left(\|\bar u\|_{C^{0,\sigma}(\bar\Omega)} + \|y_d\|_{C^{0,\sigma}(\bar\Omega)}\right) \tag{3.18}$$
$$\|\bar p - \bar p_h^s\|_{L^\infty(\Omega)} \le ch^2|\ln h|\left(\|\bar u\|_{C^{0,\sigma}(\bar\Omega)} + \|y_d\|_{C^{0,\sigma}(\bar\Omega)}\right) \tag{3.19}$$

are valid.

Proof. The proof is similar to that given in [6] for quasi-uniform meshes and $W^{2,\infty}(\Omega)$-regular solutions of the underlying boundary value problem. First, we prove assertion (3.19). One can conclude

$$\begin{aligned}\|\bar p - \bar p_h^s\|_{L^\infty(\Omega)} &= \|S^*(S\bar u - y_d) - S_h^*(S_h\bar u_h^s - y_d)\|_{L^\infty(\Omega)}\\ &\le \|(S^* - S_h^*)S\bar u\|_{L^\infty(\Omega)} + \|(S^* - S_h^*)y_d\|_{L^\infty(\Omega)}\\ &\quad + \|S_h^*S\bar u - S_h^*S_h\bar u\|_{L^\infty(\Omega)} + \|S_h^*S_h\bar u - S_h^*S_h\bar u_h^s\|_{L^\infty(\Omega)}.\end{aligned} \tag{3.20}$$

We estimate each of the four terms separately. By Theorem 2.4 it follows

$$\|(S^* - S_h^*)S\bar{u}\|_{L^\infty(\Omega)} \le ch^2|\ln h|\|S\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})} \le ch^2|\ln h|\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})} \tag{3.21}$$

since $\|S\|_{C^{0,\sigma}(\bar{\Omega})\to C^{0,\sigma}(\bar{\Omega})} \le c$. For the second term one can conclude from the same theorem

$$\|(S^* - S_h^*)Sy_d\|_{L^\infty(\Omega)} \le ch^2|\ln h|\|y_d\|_{C^{0,\sigma}(\bar{\Omega})}. \tag{3.22}$$

With the discrete Sobolev inequality (2.13) one has

$$\begin{aligned}\|S_h^*S\bar{u} - S_h^*S_h\bar{u}\|_{L^\infty(\Omega)} &\le c|\ln h|^{1/2}|S_h^*S\bar{u} - S_h^*S_h\bar{u}|_{H^1(\Omega)}\\ &\le c|\ln h|^{1/2}\|S_h^*\|_{L^2(\Omega)\to H^1(\Omega)}\|S\bar{u} - S_h\bar{u}\|_{L^2(\Omega)}\\ &\le c|\ln h|^{1/2}h^2\|\bar{u}\|_{L^2(\Omega)}\end{aligned} \tag{3.23}$$

where we have used Theorem 2.3 in the last step. Utilizing again inequality (2.13), Lemma 3.3 and Theorem 3.7 it follows for the fourth term

$$\begin{aligned}\|S_h^*S_h\bar{u} - S_h^*S_h\bar{u}_h^s\|_{L^\infty(\Omega)} &\le c|\ln h|^{1/2}|S_h^*S_h\bar{u} - S_h^*S_h\bar{u}_h^s|_{H^1(\Omega)}\\ &\le c|\ln h|^{1/2}\|S_h\bar{u} - S_h\bar{u}_h^s\|_{L^2(\Omega)}\\ &\le c|\ln h|^{1/2}\|\bar{u} - \bar{u}_h^s\|_{L^2(\Omega)}\\ &\le c|\ln h|^{1/2}h^2\left(\|\bar{u}\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)}\right).\end{aligned} \tag{3.24}$$

The estimate (3.20) yields together with (3.21) – (3.24) the assertion (3.19). Inequality (3.17) follows directly from

$$\|\bar{u} - \bar{u}_h^s\|_{L^\infty(\Omega)} \le \frac{1}{\alpha}\|\bar{p} - \bar{p}_h^s\|_{L^\infty(\Omega)}.$$

To show inequality (3.18) we conclude

$$\begin{aligned}\|y - \bar{y}_h^s\|_{L^\infty(\Omega)} &\le \|S\bar{u} - S_h\bar{u}\|_{L^\infty(\Omega)} + \|S_h\bar{u} - S_h\bar{u}_h^s\|_{L^\infty(\Omega)}\\ &\le ch^2|\ln h|\left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right),\end{aligned}$$

where we have used Theorem 2.4, Lemma 3.3 and inequality (3.17) in the last step. □

3.3 Fully discrete approach

The state equation and its adjoint are discretized according to Section 2. The control u is discretized by piecewise constant functions

$$\begin{aligned}U_h &:= \{u_h \in L^\infty(\Omega) : u_h|_T \in \mathcal{P}_0 \text{ for all } T \in T_h\},\\ U_h^{\mathrm{ad}} &:= U_h \cap U_{\mathrm{ad}}.\end{aligned}$$

Now, we introduce the discretized optimal control problem

$$\begin{aligned} J_h(\bar{u}_h) &= \min_{u_h \in U_h^{\mathrm{ad}}} J_h(u_h), \\ J_h(u_h) &:= \frac{1}{2}\|S_h u_h - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u_h\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.25}$$

Remark 3.9. *The optimal control problem (3.25) admits a unique solution $\bar{u}_h$. In the following, we use the notation $\bar{y}_h = S_h \bar{u}_h$ and $\bar{p}_h = P_h \bar{u}_h$ for the optimal discrete state and adjoint state. The variational inequality*

$$(\bar{p}_h + \nu \bar{u}_h, u_h - \bar{u}_h)_{L^2(\Omega)} \geq 0 \quad \textit{for all } u_h \in U_h^{\mathrm{ad}} \tag{3.26}$$

is necessary and sufficient for the optimality of $\bar{u}_h$.

The aim of this section is to prove error estimates of the same quality as in Theorem 3.8. For this result an assumption on the active set is needed. The optimal control $\bar{u}$ is obtained by the projection formula (3.1). This formula generates kinks in the optimal control. However, we can classify the triangles $T \in T_h$ in two sets K_1 and K_2,

$$K_1 := \bigcup_{T \in T_h:\, \bar{u} \notin V^{2,2}_{2-2\mu}(T)} T, \qquad K_2 := \bigcup_{T \in T_h:\, \bar{u} \in V^{2,2}_{2-2\mu}(T)} T. \tag{3.27}$$

Clearly, the number of triangles in K_1 grows for decreasing h. Nevertheless, the assumption

$$\operatorname{meas} K_1 \leq ch \tag{3.28}$$

is fulfilled in many practical cases.

For continuous functions f we define now the projection into the space U_h of piecewise constant functions by

$$(R_h f)(x) := f(S_T) \quad \text{if } x \in T,$$

where S_T denotes the centroid of the triangle T. Notice, that $R_h \bar{u} \in U_h^{\mathrm{ad}}$. Next, we recall two results from [1].

Theorem 3.10. *Assume that the assumption (3.28) holds. Let $\bar{u}_h$ be the solution of (3.25) on a family of meshes with grading parameter $\mu < \lambda$. Then the estimate*

$$\|\bar{u}_h - R_h \bar{u}\|_{L^2(\Omega)} \leq ch^2 \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|y_d\|_{L^\infty(\Omega)} \right) \tag{3.29}$$

holds true.

Proof. This theorem is proved in [1] under the assumption $\mu > \frac{1}{2}$, which was used in the proof of the boundedness of S_h only. The boundedness of S_h also in case of $\mu \leq 1/2$ is guaranteed by Lemma 3.3. □

Lemma 3.11. *On a mesh with grading parameter $\mu < \lambda$ the estimate*

$$(v_h, \bar{u} - R_h\bar{u})_{L^2(\Omega)} \le ch^2 \left(\|v_h\|_{L^\infty(\Omega)} + \|v_h\|_{H_0^1(\Omega)} \right) \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|\bar{y}_d\|_{L^\infty(\Omega)} \right) \qquad (3.30)$$

holds for all $v_h \in V_h$, provided that Assumption (3.28) is fulfilled.

Proof. This lemma is proved in [1]. Notice, that in that proof the condition $\mu \ge 1/2$ was not necessary. □

Next, we will apply the error estimates of Section 2 to obtain L^∞-error estimates for the optimal control problem. Before, we will derive an auxiliary result. Let us define another regularized Dirac function δ_ξ^h for a fixed point $\xi \in T_*$ with

(P1) $(\delta_\xi^h, v_h) = v_h(a) \qquad \forall v_h \in V_h$,

(P2) supp $\delta_\xi^h \subset \bar{T}_*$,

(P3) $\delta_\xi^h \in \mathcal{P}_1(T_*)$,

(P4) $\|\delta_\xi^h\|_{L^2(T_*)} = O(h_{T_*}^{-1})$.

(P5) $\|\delta_\xi^h\|_{L^\infty(\Omega)} \le c|T_*|^{-1}$

An example for a function with these properties is given in [22]. The regularized Green function z^h is defined as the solution of

$$a(v, z^h) = (\delta_\xi^h, v) \qquad \forall v \in V. \qquad (3.31)$$

Moreover, we denote by z_h^h its discrete counterpart,

$$a(v_h, z_h^h) = (\delta_\xi^h, v_h) \qquad \forall v_h \in V_h.$$

Subsequently, we need estimates of norms of the regularized Green function.

Lemma 3.12. *The norms of the regularized Green function can be estimated by*

$$\|z^h\|_{L^\infty(\Omega)} \le c|\ln h| \qquad (3.32)$$

$$\|z^h\|_{H^1(\Omega)} \le c|\ln h|^{1/2} \qquad (3.33)$$

$$\|z^h\|_{V_\beta^{2,2}(\Omega)} \le ch^{-1} \qquad (3.34)$$

where $\beta := 1 - \mu > 1 - \lambda$ is the weight corresponding to the regularity in $V_\beta^{2,2}(\Omega)$, and the grading parameter satisfies $\mu < \lambda$.

Proof. The proof of this lemma is very similar to that of Lemma 3.4. For the sake of completeness we sketch it here. Let $g(x)$ be the Green function with respect to an arbitrary point $x_+ \in \Omega$,

$$a(g, v) = v(x_+) \qquad \forall v \in V \tag{3.35}$$

According to [4] there is a constant c independent of x_+, such that

$$|g(x)| \le c(|\ln|x - x_+|| + 1).$$

Using (3.35), (3.31), the Hölder inequality and property (P4) we get

$$|z^h(x_+)| = |a(g, z^h)| = (\delta_\xi^h, g) \le \|\delta_\xi^h\|_{L^2(T_*)}\|g\|_{L^2(T_*)} \le ch_{T_*}^{-1}\|g\|_{L^2(T_*)} \tag{3.36}$$

We estimate the L^2-norm of g using polar coordinates centered in x_+,

$$\begin{aligned} h_{T_*}^{-1}\|g\|_{L^2(T_*)} &\le ch_{T_*}^{-1}\left(\int_0^{h_{T_*}} (\ln r)^2 r\mathrm{d}r\right)^{1/2} \\ &\le ch_{T_*}^{-1}\left(h_{T_*}|\ln h_{T_*}| + h_{T_*}|\ln h_{T_*}|^{1/2} + h_{T_*}\right) \\ &\le c|\ln h|. \end{aligned}$$

This yields from (3.36) the estimate (3.32).

For the proof of (3.33) we use the coercivity of the bilinear form, the definition (3.31) and property (P4) of $\delta_h(a)$,

$$\begin{aligned} \|z^h\|_{H^1(\Omega)}^2 &\le a(z^h, z^h) = (\delta^h, z^h) \le \|z^h\|_{L^\infty(\Omega)}\|\delta_h\|_{L^1(\Omega)} \\ &\le |T_*|^{1/2}\|\delta_h\|_{L^2(T_*)}\|z^h\|_{L^\infty(\Omega)} \le c\|z^h\|_{L^\infty(\Omega)}. \end{aligned}$$

With (3.32) we conclude (3.33).

The a priori estimate for the solution of the elliptic partial differential equation and the property (P5) of $\delta_h(a)$ give

$$\|z^h\|_{V_\beta^{2,2}(\Omega)} \le c\|r^\beta \delta^h\|_{L^2(\Omega)} \le c|T_*|^{-1}\|r^\beta\|_{L^2(T_*)}.$$

Since $r \le d_J$, we can continue by

$$|T_*|^{-1}\|r^\beta\|_{L^2(T_*)} \le c|T_*|^{-1/2}d_J^\beta = ch^{-1}$$

since $|T_*|^{1/2} = ch_{T_*} = chd_J^{1-\mu} = chd_J^\beta$. In the other case, $J = I$, we calculate the L^2-norm and obtain

$$|T_*|^{-1}\|r^\beta\|_{L^2(T_*)} \le c|T_*|^{-1}h_{T_*}^{\beta+1} \le ch_{T_*}^{\beta-1} = ch^{-1}$$

since $h_{T_*} = ch^{1/\mu} = ch^{1/(1-\beta)}$. Thus (3.34) is proved. □

Lemma 3.13. *The estimate*

$$\|z_h^h\|_{L^\infty(\Omega)} + \|z_h^h\|_{H_0^1(\Omega)} \le c|\ln h| \tag{3.37}$$

holds on a finite element mesh with $\mu = 1 - \beta < \lambda$.

Proof. By the triangle inequality one can conclude

$$\|z_h^h\|_{L^\infty(\Omega)} + \|z_h^h\|_{H_0^1(\Omega)} \le \|z^h\|_{L^\infty(\Omega)} + \|z^h - z_h^h\|_{L^\infty(\Omega)} + \|z_h\|_{H_0^1(\Omega)} + \|z^h - z_h^h\|_{H_0^1(\Omega)} \tag{3.38}$$

Since the meshes are optimally graded, we have

$$\|z^h - z_h^h\|_{H^1(\Omega)} \le ch|z^h|_{V_\beta^{2,2}(\Omega)}$$

and with (3.34)

$$\|z^h - z_h^h\|_{H^1(\Omega)} \le c. \tag{3.39}$$

Furthermore, we have

$$\|z^h - z_h^h\|_{L^\infty(\Omega)} \le ch|z^h|_{V_\beta^{2,2}(\Omega)}$$

and with (3.34)

$$\|z^h - z_h^h\|_{L^\infty(\Omega)} \le c. \tag{3.40}$$

Now the assertion follows from (3.38) with (3.32),(3.33), (3.39) and (3.40). □

Lemma 3.14. *The inequality*

$$\|S_h\bar{u} - S_hR_h\bar{u}\|_{L^\infty(\Omega)} \le ch^2|\ln h| \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|\bar{y}_d\|_{L^\infty(\Omega)}\right) \tag{3.41}$$

is satisfied provided that Assumption (3.28) is fulfilled.

Proof. Let $\xi \in \Omega$ be an arbitrary but fixed point. Using the definitions above, we find

$$\begin{aligned} |S_h\bar{u}(\xi) - S_hR_h\bar{u}(\xi)| &= |(\delta_\xi^h, S_h\bar{u} - S_hR_h\bar{u})| \\ &= |a(S_h\bar{u} - S_hR_h\bar{u}, z_h^h)| \\ &= |(z_h^h, \bar{u} - R_h\bar{u})|. \end{aligned}$$

Now, we can apply Lemma 3.11 and obtain

$$|S_h\bar{u}(\xi) - S_hR_h\bar{u}(\xi)| \le ch^2 \left(\|z_h^h\|_{L^\infty(\Omega)} + \|z_h^h\|_{H_0^1(\Omega)}\right) \left(\|\bar{u}\|_{L^\infty(\Omega)} + \|\bar{y}_d\|_{L^\infty(\Omega)}\right) \tag{3.42}$$

The assertion follows from (3.42) and Lemma 3.13. □

We construct now the control $\tilde{u}_h$ as a projection of the discrete adjoint state $\bar{p}_h = P_h\bar{u}_h$ to the admissible set U_{ad},

$$\tilde{u}_h = \Pi_{[u_a,u_b]}\left(-\frac{1}{\mu}\bar{p}_h\right). \tag{3.43}$$

Note, that $\tilde{u}_h$ is still piecewise linear, but in general neither in U_h nor in V_h.

The following L^∞-error estimates are new even in the case without corner singularities and uniform meshes, see [12].

Theorem 3.15. *Assume that assumption (3.28) holds. Let $\bar{y}_h$ be the associated state and $\bar{p}_h$ be the associated adjoint state to the solution $\bar{u}_h$ of (3.25) on a family of meshes with grading parameter $\mu < \lambda/2$. Further, let $\tilde{u}_h$ be the discrete control constructed in* (3.43). *Then the estimates*

$$\|\bar{y}_h - \bar{y}\|_{L^\infty(\Omega)} \le ch^2|\ln h|\left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right) \tag{3.44}$$

$$\|\bar{p}_h - \bar{p}\|_{L^\infty(\Omega)} \le ch^2|\ln h|\left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right) \tag{3.45}$$

$$\|\bar{u} - \tilde{u}_h\|_{L^\infty(\Omega)} \le ch^2|\ln h|\left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right) \tag{3.46}$$

are valid.

Proof. We start with

$$\begin{aligned}
&\|\bar{y} - \bar{y}_h\|_{L^\infty(\Omega)} = \|S\bar{u} - S_h\bar{u}_h\|_{L^\infty(\Omega)} \\
&\quad\le \|S\bar{u} - S_h\bar{u}\|_{L^\infty(\Omega)} + \|S_h\bar{u} - S_hR_h\bar{u}\|_{L^\infty(\Omega)} + \|S_hR_h\bar{u} - S_h\bar{u}_h\|_{L^\infty(\Omega)}.
\end{aligned}$$

The first term was estimated in Theorem 2.4. Lemma 3.14 delivers an inequality for the second term. Theorem 3.10 implies the estimate of the third term. Consequently, we find with the embedding $C^{0,\sigma}(\bar{\Omega}) \hookrightarrow L^\infty(\Omega)$

$$\begin{aligned}
\|\bar{y} - \bar{y}_h\|_{L^\infty(\Omega)} &\le ch^2|\ln h|\left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})} + \|y_d\|_{L^\infty(\Omega)}\right) \\
&\le ch^2|\ln h|\left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right),
\end{aligned}$$

i.e., (3.44). The second inequality can be obtained similarly,

$$\begin{aligned}
\|\bar{p} - \bar{p}_h\|_{L^\infty(\Omega)} &= \|S^*(\bar{y} - y_d) - S_h^*(\bar{y}_h - y_d)\|_{L^\infty(\Omega)} \\
&\le \|S^*(\bar{y} - y_d) - S_h^*(\bar{y} - y_d)\|_{L^\infty(\Omega)} + \|S_h^*(\bar{y} - \bar{y}_h)\|_{L^\infty(\Omega)} \\
&\le ch^2|\ln h|\|\bar{y} - y_d\|_{C^{0,\sigma}(\bar{\Omega})} + ch^2|\ln h|\left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right) \\
&\le ch^2|\ln h|\left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})} + \|y_d\|_{C^{0,\sigma}(\bar{\Omega})}\right)
\end{aligned}$$

by means of (3.44) and Theorem 2.4. To prove the third inequality we use, that the projection operator $\Pi_{[a,b]}$ is Lipschitz continuous with constant 1 from $L^\infty(\Omega)$ to $L^\infty(\Omega)$. Therefore, we get

$$\begin{aligned}\nu\|\bar u - \tilde u_h\|_{L^\infty(\Omega)} &= \nu \left\| \Pi_{[a,b]}\left(-\frac{1}{\nu}\bar p\right) - \Pi_{[a,b]}\left(-\frac{1}{\nu}\bar p_h\right)\right\|_{L^\infty(\Omega)} \\ &\le \|\bar p - \bar p_h\|_{L^\infty(\Omega)} \\ &\le ch^2 |\ln h| \left(\|\bar u\|_{C^{0,\sigma}(\bar\Omega)} + \|y_d\|_{C^{0,\sigma}(\bar\Omega)}\right).\end{aligned}$$

where we used (3.46) in the last step. The superconvergence result is proved. □

4 Numerical example

In this section we illustrate our theoretical findings for the fully discrete approach by some numerical tests. Therefore we consider the optimal control problem (1.1)–(1.6) with $L = -\Delta$ and the first-order optimality system

$$\begin{aligned} -\Delta y &= u + f \quad \text{in } \Omega, \quad y = 0 \quad \text{on } \partial\Omega, \\ -\Delta p &= y - y_d \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \partial\Omega, \\ u &= \Pi_{[a,b]}\left(-\frac{1}{\nu}p\right). \end{aligned}$$

The data y_d and f are chosen such that the exact solution is given as

$$\begin{aligned} \bar y(r,\varphi) &= (r^\lambda - r^\alpha)\sin\lambda\varphi, \\ \bar p(r,\varphi) &= \nu(r^\lambda - r^\beta)\sin\lambda\varphi. \end{aligned}$$

We set $\alpha = \beta = \frac{5}{2}$ and $\mu = 10^{-3}$. To evaluate the maximum norm of the error we used not only grid points but also the nodes of a high order quadrature formula of degree 19 implemented in the program package MooNMD [8]. In the following we study the example in a convex and a non-convex domain.

4.1 Example in a convex domain

The domain Ω is defined as

$$\Omega = \left\{(r\cos\varphi, r\sin\varphi)^T : 0 < r < 1, 0 < \varphi < \frac{3}{4}\pi\right\}$$

and therefore $\lambda = \frac{4}{3}$. Table 1 shows the computed errors $\|\bar u - \tilde u_h\|_{L^\infty(\Omega)}$ for quasi-uniform meshes and for graded meshes with $\mu = 0.6 < \frac{\lambda}{2}$. While a convergence rate of about λ can be observed for $\mu = 1$, the approximation order is slightly smaller than 2 on the graded meshes. So mesh grading improves the convergence rate for the L^∞-error also in the case of a corner with an interior angle between $\frac{\pi}{2}$ and π.

Figure 2: Convex domain with a quasi-uniform mesh ($\mu = 1$) and a graded mesh with $\mu = 0.6$.

	$\mu = 0.6$		$\mu = 1$	
ndof	value	rate	value	rate
51	3.02e−02		4.63e−02	
176	1.19e−02	1.50	1.88e−02	1.45
651	4.12e−03	1.62	7.57e−03	1.39
2501	1.42e−03	1.58	3.02e−03	1.36
9801	4.22e−04	1.78	1.20e−03	1.35
38801	1.15e−04	1.89	4.79e−04	1.34
74482	6.11e−05	1.94	3.09e−04	1.34
154401	3.01e−05	1.95	1.90e−04	1.34

Table 1: L^∞-error of the computed control $\tilde{u}_h$ in a convex domain

4.2 Example in a non-convex domain

As second example we set Ω as

$$\Omega = \left\{ (r\cos\varphi, r\sin\varphi)^T : 0 < r < 1, 0 < \varphi < \frac{3}{2}\pi \right\}.$$

This means $\lambda = \frac{2}{3}$. In Table 2 one can find the computed errors $\|\bar{u} - \tilde{u}_h\|_{L^\infty(\Omega)}$ on different meshes with $\mu = 0.3 < \frac{\lambda}{2}$, $\mu = 0.6 < \lambda$ and $\mu = 1.0$. For meshes with grading parameter $\mu < \frac{\lambda}{2}$ one can see the predicted convergence rate slightly smaller than 2. Further one can observe, that a mesh grading parameter $\mu \in \left(\frac{\lambda}{2}, \lambda\right)$ yields only a suboptimal convergence rate $\frac{\lambda}{\mu} = \frac{10}{9}$ for the L^∞-error. Notice, that such a mesh grading was enough to get the optimal convergence of second order for the L^2-error (see [1]). If no mesh grading is performed ($\mu = 1$) one can observe a convergence rate of about λ.

In Figure 3 the distribution of the L^∞-error on a uniform and on an appropriately graded mesh is shown. The mesh grading significantly reduces the error near the edge.

	$\mu = 0.3$		$\mu = 0.6$		$\mu = 1$	
ndof	value	rate	value	rate	value	rate
125	1.63e−01		8.18e−02		2.19e−01	
286	7.61e−01	1.20	3.77e−02	1.23	1.34e−01	0.78
1071	2.30e−02	1.81	1.73e−02	1.18	6.67e−02	1.06
4141	7.49e−03	1.66	7.99e−03	1.14	4.15e−02	0.70
16281	1.97e−03	1.95	3.70e−03	1.13	2.60e−02	0.68
25351	1.29e−03	1.92	2.88e−03	1.12	2.24e−02	0.68
39501	8.29e−04	1.98	2.25e−03	1.12	1.93e−02	0.68
100701	3.34e−04	1.95	1.33e−03	1.12	1.41e−02	0.67

Table 2: L^∞-error of the computed control $\tilde{u}_h$ in a non-convex domain

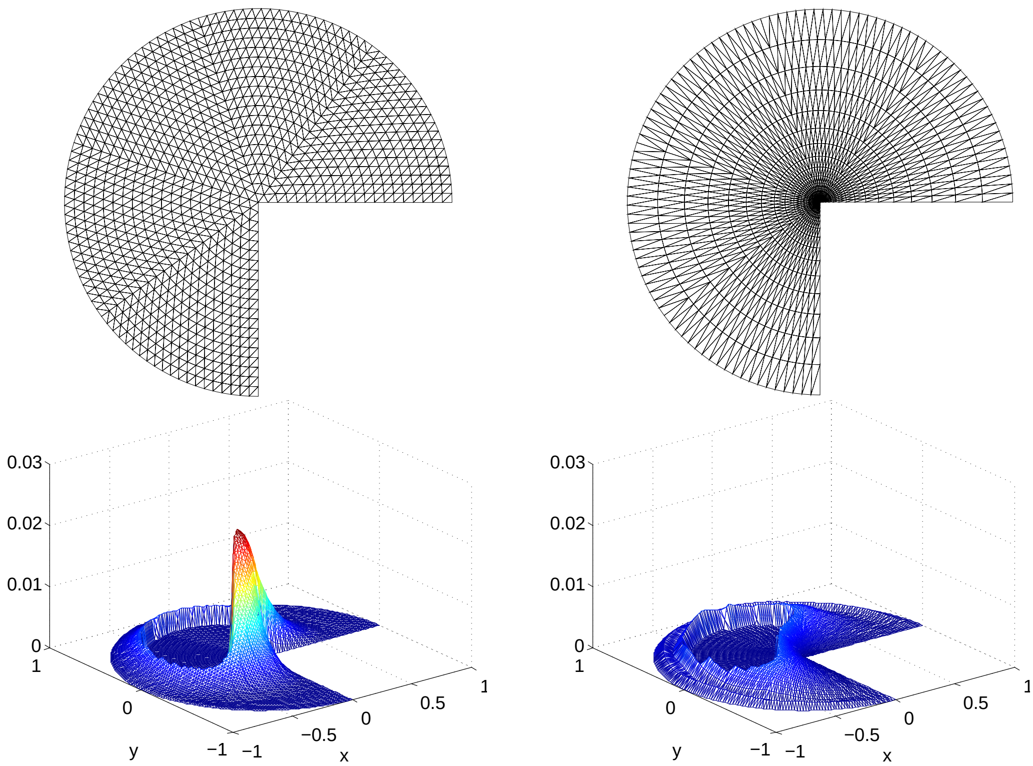

Figure 3: Error distribution on a uniform (left) and a graded mesh with $\mu = 0.3$.

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References

[1] Th. Apel, A. Rösch, and G. Winkler. Optimal control in non-convex domains: a priori discretization error estimates. *Calcolo*, 44:137–158, 2007.

[2] Th. Apel, A.-M. Sändig, and J. R. Whiteman. Graded mesh refinement and error estimates for finite element solutions of elliptic boundary value problems in non-smooth domains. *Math. Methods Appl. Sci.*, 19:63–85, 1996.

[3] I. Babuska, R. B. Kellogg, and J. Pitkäranta. Direct and inverse error estimates for finite elements with mesh refinements. *Numer. Math.*, 33:447–471, 1979.

[4] G. Dolzmann and S. Müller. Estimates for Green's matrices of elliptic systems by L^p theory. *Manuscripta Math.*, 88:261–273, 1995.

[5] J. Frehse and R. Rannacher. Eine L^1-Fehlerabschätzung für diskrete Grundlösungen in der Methode der finiten Elemente. In J. Frehse, R. Leis, and R. Schaback, editors, *Finite Elemente. Tagungsband des Sonderforschungsbereichs 72*, volume 89 of *Bonner Mathematische Schriften*, pages 92–114, Bonn, 1976.

[6] M. Hinze. A variational discretization concept in control constrained optimization: The linear-quadratic case. *Comput. Optim. Appl.*, 30:45–61, 2005.

[7] V.P. Il'in. Some inequalities in function spaces and their application to the study of the convergence of variational processes. *Trans. Amer. Math. Soc.*, 81:1–66, 1969.

[8] V. John and G. Matthies. Moonmd–a program package based on mapped finite element methods. *Computing and Visualization in Science*, 6:163–169, 2004.

[9] V. A. Kondrat'ev. Boundary value problems for elliptic equations on domains with conical or angular points. *Trudy Moskov. Mat. Obshch.*, 16:209–292, 1967. In Russian.

[10] V. A. Kozlov, V. G. Maz'ya, and J. Roßmann. *Spectral Problems Associated with Corner Singularities of Solutions to Elliptic Equations*. American Mathematical Society, Providence, RI, 2001.

[11] V.G. Maz'ya and B. A. Plamenevskii. Weighted spaces with nonhomogeneous norms and boundary value problems in domains with conical points. *Trans. Amer. Math. Soc.*, 123:89–107, 1984.

[12] C. Meyer and A. Rösch. Superconvergence properties of optimal control problems. *SIAM J. Control Optim.*, 43:970–985, 2004.

[13] C. Meyer and A. Rösch. L^∞-estimates for approximated optimal control problems. *SIAM J. Control Optim.*, 44:1636–1649, 2005.

[14] F. Natterer. Über die punktweise Konvergenz Finiter Elemente. *Numer. Math.*, 25:67–77, 1975.

[15] L. A. Oganesyan and L. A. Rukhovets. *Variational-difference methods for the solution of elliptic equations.* Izd. Akad. Nauk Armyanskoi SSR, Jerevan, 1979. In Russian.

[16] L. A. Oganesyan, L. A. Rukhovets, and V. Ja. Rivkind. *Variational-difference methods for solving elliptic equations, Part II*, volume 8 of *Differential equations and their applications.* Izd. Akad. Nauk Lit. SSR, Vilnius, 1974. In Russian.

[17] G. Raugel. *Résolution numérique de problèmes elliptiques dans des domaines avec coins.* PhD thesis, Université de Rennes, 1978.

[18] A.-M. Sändig. Error estimates for finite element solutions of elliptic boundary value problems in non-smooth domains. *Z. Anal. Anwend.*, 9(2):133–153, 1990.

[19] A. H. Schatz and L. B. Wahlbin. Interior maximum norm estimates for finite element methods. *Math. Comp.*, 31:414–442, 1977.

[20] A. H. Schatz and L. B. Wahlbin. Maximum norm estimates in the finite element method on plane polygonal domains. Part 1. *Math. Comp.*, 32(141):73–109, 1978.

[21] A. H. Schatz and L. B. Wahlbin. Maximum norm estimates in the finite element method on plane polygonal domains. Part 2: Refinements. *Math. Comp.*, 33(146):465–492, 1979.

[22] R. Scott. Finite element convergence for singular data. *Numer. Math.*, 21(4):317–327, 1973.

[23] R. Scott. Optimal L^∞ estimates for the finite element method on irregular meshes. *Math. Comp.*, 30:681–697, 1976.

[24] W. L. Wendland. *Elliptic systems in the plane.* Pitman, London, 1979.