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A priori mesh grading for an elliptic problem with Dirac right-hand side

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Abstract The fundamental solution of the Laplace equation in two dimensions is not contained in the Sobolev space $H^1(\Omega)$ such that finite element error estimates are non-standard and quasi-uniform meshes are inappropriate. By using graded meshes L^2-error estimates of almost optimal order are shown.

As a by-product, we show for the Poisson equation with a right-hand side in L^2 that appropriate mesh refinement near some interior point diminishes the error in this point by nearly one order.

Key Words Dirac measure, fundamental solution, finite element method, graded mesh, error estimate

AMS subject classification 65N30, 65N15

1 Introduction

In this paper we consider the finite element solution of the elliptic boundary value problem

$$-\Delta u = \delta_{x_0} \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a convex, polygonal domain and δ_{x_0} denotes the Dirac measure concentrated at $x_0 \in \text{int}(\Omega)$ such that $\text{dist}(x_0, \partial\Omega) > 0$. We restrict the consideration to convex domains in order to avoid additional mesh refinement to treat corner singularities.

Problems of this type occur in the simulation of field problems including point forces in linear elasticity or point charges in electrical field calculations. Furthermore, the

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application of the dual weighted residual method for estimating pointwise errors in a finite element discretization of a partial differential equation with smooth data leads to a problem of type (1.1), see [4]. Dirac measure terms can also be contained in the right hand side of the adjoint problem in optimal control of elliptic partial differential equations with state constraints [7]. As a last example where point sources occur, we mention parameter identification problems with pointwise measurements [13].

For applying the finite element method, problem (1.1) is non-standard since there is no $H^1(\Omega)$-solution. Hence the terms *solution* and *finite element solution* have to be defined carefully. The error analysis cannot start with the $H^1(\Omega)$-error such that also the Aubin–Nitsche method for obtaining an $L^2(\Omega)$-error estimate cannot be applied without modification.

To the best of our knowledge, a priori discretization error estimates for such a type of problem were proved in the case of quasi-uniform meshes only. Babuška [3] proved general error estimates and specified for Dirac right hand side and a two-dimensional smooth domain the almost optimal convergence order $h^{1-\varepsilon}$ for arbitrary $\varepsilon > 0$. Scott [14] considered differential equations of order $2m$ in smooth d-dimensional domains and finite elements polynomial degree $k-1$. He proved for the error in the H^s-norm, $2m-k \leq s \leq 2m-\frac{d}{2}$, $k \geq m$, the convergence order $2m-\frac{d}{2}-s$, i. e., order $2-\frac{d}{2}$ in the L^2-norm for our problem. Casas [6] proved the same result for polygonal or polyhedral domains and general regular Borel measures on the right-hand side using a different technique for the proof.

Our contribution is the investigation of locally refined meshes. We introduce graded meshes with respect to the point $x_0 \in \Omega$ which allow to prove two results. The first one is an intermediate step towards our final result but also of independent interest: We investigate the influence of the mesh refinement near x_0 on the finite element approximation of the solution of the Poisson equation with a right-hand side in $L^2(\Omega)$ and prove a convergence rate of $h^2 |\ln h|^{3/2}$ for the error in the point x_0. Note that in this case the solution is in general in $H^2(\Omega)$ but not in $W^{2,\infty}(\Omega)$ such that one can only expect first order convergence in $L^\infty(\Omega)$ on uniform meshes. This means, that the mesh grading improves the approximation quality in the point x_0 significantly. From this result we can derive the second result, an L^2-error estimate of order $h^2 |\ln h|^{3/2}$ for the finite element approximation of problem (1.1).

The outline of the paper is as follows. In the next short section we introduce the graded meshes and state the results. We prove the main result on the basis of Theorem 2.1 whose proof is postponed to the third section. There, we formulate a couple of auxiliary results and conclude the result of Theorem 2.1 by using techniques developed by Frehse and Rannacher [8]. In Section 4 we illustrate our theoretical findings by a numerical test.

Remark 1.1. *In the case when problem* (1.1) *appears as the adjoint problem of an optimal control problem with state constraints, the location of x_0 is generally unknown such that one has to rely on adaptive procedures [5, 9]. The graded meshes and error estimates developed in this paper show that an L^2-error of almost $\mathcal{O}(N^{-1})$, where N is the number of elements, can be achieved which gives the target quality for any adaptive calculation.*

Remark 1.2. *A first error estimator which is proved to be reliable and efficient for elliptic problems with Dirac right hand side was introduced by Araya, Behrens and Rodríguez [2]. Numerical experiments with unstructured adaptive meshes yield the optimal convergence $\mathcal{O}(N^{-1})$ in the L^2-norm without the logarithmic factor. The experiments cover the case when x_0 is or is not a mesh point.*

We point out that the work in [2] is based on a known location of x_0. In our target application indicated in Remark 1.1 this information is not available.

We end this section with the explanation of some notation. As usual, we denote by $W^{s,p}(\Omega)$ the Sobolev spaces and write $H^s(\Omega)$ for $W^{s,2}(\Omega)$. The scalar product in $L^2(\Omega)$ is denoted by $(.,.)$. C is a generic positive constant independent of h, and the notation $a \sim b$ means the existence of two constants c and C such that $ca \le b \le Ca$.

2 Main results

As mentioned in the introduction problem (1.1) does not have an $H^1(\Omega)$-solution. Therefore we follow [2] and consider the solution u in the space

$$W_0^{1,q}(\Omega) := \{v \in W^{1,q}(\Omega) \,:\, v = 0 \text{ on } \partial\Omega \text{ in the sense of } L^q(\partial\Omega)\},$$

$q \in [1,2)$, defined via

$$(\nabla u, \nabla v) = v(x_0) \quad \forall v \in W_0^{1,q'}(\Omega)$$

where $q' > 2$ satisfies $1/q + 1/q' = 1$.

For the approximate solution we introduce a family of graded triangulations $\mathcal{T}_h$ of Ω, where h is the discretization parameter. With r_T denoting the distance of an element $T \in \mathcal{T}_h$ to x_0, we set the element sizes according to

$$h_T \sim \begin{cases} h^2 & \text{if } r_T = 0, \\ h r_T^{1/2} & \text{if } r_T > 0. \end{cases} \tag{2.1}$$

Notice that the number of elements of such a triangulation is of order h^{-2}, see, for example, [1]. The finite element space is then defined by

$$V_h := \{v_h \in C(\bar{\Omega}) \,:\, v_h|_T \in \mathcal{P}_1(T) \text{ for all } T \in \mathcal{T}_h \text{ and } v_h = 0 \text{ on } \partial\Omega\}$$

where $\mathcal{P}_1(T)$ denotes the space of polynomials on T of degree at most one. The finite element solution $u_h \in V_h$ of (1.1) satisfies

$$(\nabla u_h, \nabla v_h) = v_h(x_0) \quad \forall v_h \in V_h. \tag{2.2}$$

The second problem considered is the Poisson problem with a right-hand side $f \in L^2(\Omega)$,

$$-\Delta z = f \text{ in } \Omega, \quad z = 0 \text{ on } \partial\Omega, \tag{2.3}$$

which has no connection to the point $x_0 \in \mathrm{int}(\Omega)$ from the initial problem. In the following theorem we state that the proposed mesh grading (2.1) with respect to the point $x_0 \in \mathrm{int}(\Omega)$ yields an improvement in the convergence rate in the error in this point. Notice that one would expect only a convergence rate of h on quasi-uniform meshes.

Theorem 2.1. *Let $f \in L^2(\Omega)$ and z_h be the finite element solution of problem* (2.3) *using the finite element space V_h defined above on a mesh that is graded according to condition* (2.1)*. Then the a priori estimate*

$$|(z - z_h)(x_0)| \le ch^2 |\ln h|^{3/2} \|z\|_{H^2(\Omega)}$$

holds.

The proof of the theorem is postponed to Section 3. From this theorem one can conclude the main result of the paper, an L^2-error estimate for problem (1.1).

Corollary 2.1. *Let u be the solution of* (1.1) *and u_h its finite element approximation defined via* (2.2) *on a family of meshes that are graded according to condition* (2.1)*. Then the a priori estimate*

$$\|u - u_h\|_{L^2(\Omega)} \le ch^2 |\ln h|^{3/2}$$

holds.

Proof. Denoting the error by $e := u - u_h$, we define the function $v \in H_0^1(\Omega)$ as the solution of

$$(\nabla v, \nabla \phi) = (e, \phi) \quad \forall \phi \in H_0^1(\Omega),$$

i.e. the weak solution of the boundary value problem

$$-\Delta v = e \;\text{ in } \Omega, \quad v = 0 \;\text{ on } \partial\Omega.$$

Its finite element approximation v_h is defined by

$$(\nabla v_h, \nabla \phi_h) = (e, \phi_h) \quad \forall \phi_h \in V_h.$$

With these auxiliary quantities we can estimate by utilizing Theorem 2.1

$$\begin{aligned} \|u - u_h\|^2_{L^2(\Omega)} &= \|e\|^2 = (e, u) - (e, u_h) \\ &= (\nabla v, \nabla u) - (\nabla v, \nabla u_h) \\ &= v(x_0) - v_h(x_0) = (v - v_h)(x_0) \\ &\le ch^2 |\ln h|^{3/2} \left\|\nabla^2 v\right\|_{L^2(\Omega)} \\ &\le ch^2 |\ln h|^{3/2} \|e\|_{L^2(\Omega)} . \end{aligned}$$

Dividing this inequality by $\|u - u_h\|_{L^2(\Omega)}$ gives the desired result. □

3 Proof of Theorem 2.1

We state a couple of auxiliary results in the forthcoming lemmas. At the end of the section we use these results to prove Theorem 2.1.

We split the domain Ω into the sets

$$\Omega_0 = \bigcup_{r_T = 0} T \text{ and } \Omega_1 = \Omega \setminus \Omega_0$$

and choose an element $T^* \in \Omega_0$. Its diameter is $h_* \sim h^2$. A weight function $\sigma : \Omega \to \mathbb{R}$ is defined by

$$\sigma(x) := \left(|x - x_0|^2 + \kappa^2 h_*^2 \right)^{1/2} \tag{3.1}$$

with some $\kappa > 0$. We collect some properties of σ which follow from basic calculations in the following lemma.

Lemma 3.1. *For the function σ defined in* (3.1) *the inequalities*

$$\begin{aligned} |\sigma| + |\nabla\sigma| &\leq c \\ \left|\nabla^2\sigma\right| &\leq c\sigma^{-1} \\ \sigma^{-1}(x) &\leq \begin{cases} ch_*^{-1} & \text{if } x \in \{T \in \mathcal{T}_h \,:\, r_T = 0\}, \\ cr_T^{-1} & \text{if } x \in \{T \in \mathcal{T}_h \,:\, r_T > 0\} \end{cases} \end{aligned} \tag{3.2}$$

are valid.

For functions with elementwise L^2-regularity we introduce the norm

$$\|v\|_h := \left(\sum_{T \in \mathcal{T}_h} \|v\|_{L^2(T)}^2 \right)^{1/2}.$$

The nodal interpolant of a function $v \in C(\bar{\Omega})$ is denoted by $\mathcal{I}_h v$. We begin our considerations with an estimate of a weighted interpolation error in the following lemma.

Lemma 3.2. *For any function v from the set*

$$\{v \in H^1(\Omega) \,:\, v \in H^2(T) \; \forall T \in \mathcal{T}_h\}$$

the estimate

$$\left\| \sigma^{-1/2} \nabla(v - \mathcal{I}_h v) \right\|_{L^2(\Omega)} \leq ch \left\| \nabla^2 v \right\|_h$$

holds on meshes of type (2.1). *For functions $v \in H^2(\Omega)$ this results in*

$$\left\| \sigma^{-1/2} \nabla(v - \mathcal{I}_h v) \right\|_{L^2(\Omega)} \leq ch \left\| \nabla^2 v \right\|_{L^2(\Omega)}.$$

Proof. One can calculate by using (3.2)

$$\begin{aligned} \left\| \sigma^{-1/2} \nabla(v - \mathcal{I}_h v) \right\|_{L^2(\Omega)}^2 &= \sum_{T \subset \Omega_0} \int_T \sigma^{-1} |\nabla(v - \mathcal{I}_h v)|^2 + \sum_{T \subset \Omega_1} \int_T \sigma^{-1} |\nabla(v - \mathcal{I}_h v)|^2 \\ &\leq \sum_{T \subset \Omega_0} ch_*^{-1} h_*^2 \left\| \nabla^2 v \right\|_{L^2(T)}^2 + \sum_{T \subset \Omega_1} cr_T^{-1} h_T^2 \left\| \nabla^2 v \right\|_{L^2(T)}^2 \\ &\leq \sum_{T \subset \Omega} ch^2 \left\| \nabla^2 v \right\|_{L^2(T)}^2. \end{aligned}$$

This proves the assertion. $\square$

Lemma 3.3. *For any function $v \in H^2(\Omega)$ the inequality*

$$\|\nabla(v - \mathcal{I}_h v)\|_{L^2(\Omega)} \le c \left\|\sigma \nabla^2 v\right\|_{L^2(\Omega)}$$

holds provided the mesh is graded according to (2.1)*.*

Proof. With the help of the function σ we can estimate the element size on the two subdomains. On Ω_0 there is $\sigma(x) \ge \kappa h_*$ and it follows

$$h_*^2 \le c\sigma^2(x) \qquad \forall x \in \Omega_0. \tag{3.3}$$

On Ω_1 one has $\sigma(x) \ge r_T$ and $\sigma(x) \ge ch_*$ and since $h_T \sim h r_T^{1/2}$ it follows $h_T^2 \sim h^2 r_T \sim h_* r_T$ and one can conclude

$$h_T^2 \le c\sigma^2(x) \qquad \forall x \in \Omega_1. \tag{3.4}$$

Now we can estimate

$$\|\nabla(v - \mathcal{I}_h v)\|_{L^2(\Omega)}^2 \le c \sum_T \int_T h_T^2 \left|\nabla^2 v\right|^2 = c \sum_{T \subset \Omega_0} \int_T h_*^2 \left|\nabla^2 v\right|^2 + c \sum_{T \subset \Omega_1} \int_T h_T^2 \left|\nabla^2 v\right|^2 .$$

With the estimates (3.3), (3.4) one can continue with

$$\|\nabla(v - \mathcal{I}_h v)\|_{L^2(\Omega)}^2 \le c \sum_T \int_T \sigma^2 \left|\nabla^2 v\right|^2 \le c \left\|\sigma \nabla^2 v\right\|_{L^2(\Omega)}^2 ,$$

and the assertion is proved. $\square$

Lemma 3.4. *Let the function $z \in H^2(\Omega)$ be the solution of boundary value problem* (2.3) *with a given right-hand side $f \in L^2(\Omega)$. Then the estimate*

$$\left\|\sigma \nabla^2 z\right\|_{L^2(\Omega)} \le c \left|\ln h\right| \|\sigma f\|_{L^2(\Omega)}$$

holds, where σ is the weight function defined in (3.1)*.*

Proof. Set $\xi := x - x_0$ and denote by ξ_1, ξ_2 its components. By the chain rule it holds

$$\left\|\xi_i \nabla^2 z\right\|_{L^2(\Omega)} \le \left\|\nabla^2 (\xi_i z)\right\|_{L^2(\Omega)} + \|\nabla z\|_{L^2(\Omega)} .$$

With the definition of σ this yields

$$\begin{aligned}
\left\|\sigma \nabla^2 z\right\|_{L^2(\Omega)}^2 &= \sum_{i=1}^2 \left\|\xi_i \nabla^2 z\right\|_{L^2(\Omega)}^2 + \kappa^2 h_*^2 \left\|\nabla^2 z\right\|_{L^2(\Omega)}^2 \\
&\le \sum_{i=1}^2 \left(\left\|\nabla^2 (\xi_i z)\right\|_{L^2(\Omega)}^2 + \|\nabla z\|_{L^2(\Omega)}^2 \right) + c h_*^2 \|\Delta z\|_{L^2(\Omega)}^2 .
\end{aligned}$$

With the use of $h_* \le c\sigma$ we continue

$$\begin{aligned}
\left\|\sigma\nabla^2 z\right\|^2_{L^2(\Omega)} &\le \sum_{i=1}^{2} \|\Delta(\xi_i z)\|^2_{L^2(\Omega)} + c\,\|\nabla z\|^2_{L^2(\Omega)} + c\,\|\sigma\Delta z\|^2_{L^2(\Omega)} \\
&\le c\sum_{i=1}^{2} \|\xi_i \Delta z\|^2_{L^2(\Omega)} + c\,\|\nabla z\|^2_{L^2(\Omega)} + c\,\|\sigma f\|^2_{L^2(\Omega)} \\
&\le c\,\|\sigma\Delta z\|^2_{L^2(\Omega)} + c\,\|\nabla z\|^2_{L^2(\Omega)} + c\,\|\sigma f\|^2_{L^2(\Omega)} \\
&\le c\,\|\sigma f\|^2_{L^2(\Omega)} + c\,\|\nabla z\|^2_{L^2(\Omega)}\,.
\end{aligned} \tag{3.5}$$

where we have used inequality (3.2) and the definition (2.3) of z. It remains to show that $\|\nabla z\|_{L^2(\Omega)} \le |\ln h|\,\|\sigma f\|_{L^2(\Omega)}$. Following an argument taken from [11] we consider

$$\|\nabla z\|^2_{L^2(\Omega)} = |(\Delta z, z)| \le \|\sigma\Delta z\|_{L^2(\Omega)} \left\|\sigma^{-1} z\right\|_{L^2(\Omega)} = \|\sigma f\|_{L^2(\Omega)} \left\|\sigma^{-1} z\right\|_{L^2(\Omega)}\,. \tag{3.6}$$

The last factor will be estimated by using its representation in polar coordinates (r, θ) with respect to x_0. In the following we use the observation

$$\sigma(r) = \left(r^2 + \kappa^2 h_*^2\right)^{\frac{1}{2}} \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}r}(\ln\sigma(r) - \ln\sigma(0)) = \frac{r}{\sigma^2} \tag{3.7}$$

and the inequality

$$\left|\frac{\ln\sigma(r) - \ln\sigma(0)}{r}\right| \le \frac{c}{\sigma}\,|\ln h|\,, \tag{3.8}$$

which is proved later. Furthermore for simplicity of notation we replace the integration domain Ω by a disc of some radius R that contains Ω. We continue the function z with $z = 0$ outside the domain Ω such that this extension of the domain does not change the value of any quantities involved. With the observation (3.7), partial integration with respect to the radius r, and the estimate (3.8) one can conclude

$$\begin{aligned}
\left\|\sigma^{-1} z\right\|^2_{L^2(\Omega)} &= \int_\Omega \sigma^{-2} z^2 = \int_0^{2\pi}\!\!\int_0^R r\sigma^{-2} z^2 \,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi}\!\!\int_0^R \frac{|\ln\sigma(r) - \ln\sigma(0)|}{r} r\, 2z\partial_r z \,\mathrm{d}r\,\mathrm{d}\theta \\
&\le \int_0^{2\pi}\!\!\int_0^R \frac{c}{\sigma}\,|\ln h|\, r z\partial_r z \,\mathrm{d}r\,\mathrm{d}\theta \\
&\le c\,|\ln h| \int_0^{2\pi}\!\!\int_0^R \sigma^{-1} r z\,|\nabla z|\,\mathrm{d}r\,\mathrm{d}\theta \\
&\le c\,|\ln h| \left\|\sigma^{-1} z\right\|_{L^2(\Omega)} \|\nabla z\|_{L^2(\Omega)}\,.
\end{aligned}$$

Dividing by $\left\|\sigma^{-1}z\right\|_{L^2(\Omega)}$ yields

$$\left\|\sigma^{-1}z\right\|_{L^2(\Omega)} \le c\,|\ln h|\,\|\nabla z\|_{L^2(\Omega)}\,.$$

Inserting this into equation (3.6) and dividing by $\|\nabla z\|_{L^2(\Omega)}$ yields

$$\|\nabla z\|_{L^2(\Omega)} \le c\,|\ln h|\,\|\sigma f\|_{L^2(\Omega)}$$

and thus with (3.5) the claim of the lemma.

It remains to prove inequality (3.8). To this end, we distinguish the cases $r > h_*$ and $r \le h_*$ and begin with the case $r > h_*$. Since $\sigma(r)$ is strictly monotone $|\ln\sigma(r)|$ takes its maximum at the left or right boundary of $[0,R]$. The values are

$$|\ln\sigma(0)| = |\ln(\kappa h_*)| \le c\,|\ln h|\,, \tag{3.9}$$

$$|\ln\sigma(R)| = \left|\ln\sqrt{R^2+\kappa^2h_*^2}\right| \le c\ln R, \tag{3.10}$$

thus

$$|\ln\sigma(r) - \ln\sigma(0)| \le 2\max_{0\le r\le R}|\ln\sigma(r)| \le c\,|\ln h|$$

for sufficiently small h. Since it is $1/r \le c/\sigma$ the inequality (3.8) is proved.

For the case $r \le h_*$ we can conclude by the mean value theorem

$$\left|\frac{\ln\sigma(r)-\ln\sigma(0)}{r}\right| \le \max_{0\le s\le h_*}\left|(\ln\sigma)'(s)\right| = \max_{0\le s\le h_*}\frac{s}{\sigma(s)^2}.$$

As the last function is monotonically increasing on $[0,h_*]$ it takes its maximum at the end of the interval. This means by using $\sigma(r) \le (1+\kappa^2)^{1/2}h_*$

$$\left|\frac{\ln\sigma(r)-\ln\sigma(0)}{r}\right| \le \frac{h_*}{(1+\kappa^2)h_*^2} \le c\frac{1}{\sigma}$$

and inequality (3.8) is also proved in this case. $\square$

For our further considerations we introduce a regularized Dirac function, see, e.g. [8],

$$\delta^h := \begin{cases} |T^*|^{-1}\operatorname{sign}(u-u_h) & \text{in } T^*, \\ 0 & \text{elsewhere.}\end{cases}$$

Notice, that $\delta^h \in L^2(\Omega)$. The corresponding regularized Green function $g^h \in H^2(\Omega)$ is defined by

$$-\Delta g^h = \delta^h \text{ in } \Omega, \qquad g^h = 0 \text{ on } \partial\Omega. \tag{3.11}$$

Besides, we define the function g_h^h as Ritz projection of g^h onto V_h, i.e.,

$$(\nabla g_h^h, \nabla\phi_h) = (\nabla g^h, \nabla\phi_h) \quad \forall\phi_h \in V_h. \tag{3.12}$$

Lemma 3.5. *For the regularized Green function g^h defined in* (3.11) *the estimate*

$$\left\|\sigma\nabla^2 g^h\right\|_{L^2(\Omega)} \le c\, |\ln h|^{1/2}$$

holds.

Proof. The assertion follows from setting $\rho = h_*$ in [8, Theorem B4]. In this paper, a $C^{1,1}$-domain Ω is considered but this assumption is not necessary for the result of this lemma, see also [12, Lemma 3.7]. □

Lemma 3.6. *For the regularized Green function g^h and its Ritz projection g_h^h defined in* (3.11) *and* (3.12)*, respectively, the inequality*

$$\left\|\sigma^{-1}(g^h - g_h^h)\right\|_{L^2(\Omega)} \le c\,|\ln h|^{3/2}$$

is satisfied.

Proof. We introduce the abbreviation $e_g := g^h - g_h^h$ and consider the auxiliary equation

$$-\Delta z = \frac{\sigma^{-2}e_g}{\|\sigma^{-1}e_g\|_{L^2(\Omega)}} \text{ in } \Omega, \qquad z = 0 \text{ on } \partial\Omega.$$

Its weak form can be written as

$$(\nabla z, \nabla\varphi) = \frac{(\sigma^{-1}e_g, \sigma^{-1}\varphi)}{\|\sigma^{-1}e_g\|_{L^2(\Omega)}}.$$

The choice $\varphi = e_g$ yields

$$\left\|\sigma^{-1}e_g\right\|_{L^2(\Omega)} = (\nabla e_g, \nabla z) = (\nabla e_g, \nabla(z - \mathcal{I}_h z)) \le \|\nabla e_g\|_{L^2(\Omega)}\, \|\nabla(z - \mathcal{I}_h z)\|_{L^2(\Omega)}\,. \tag{3.13}$$

For the first term of the right-hand side we use Lemma 3.3 with the choice $v = g^h$ and conclude

$$\|\nabla e_g\|_{L^2(\Omega)} \le c\left\|\nabla(g^h - \mathcal{I}_h g^h)\right\|_{L^2(\Omega)} \le c\left\|\sigma\nabla^2 g^h\right\|_{L^2(\Omega)} \le c\,|\ln h|^{1/2} \tag{3.14}$$

where we have used Lemma 3.5 in the last step. For the second term on the right-hand side of inequality (3.13) we write with Lemmas 3.3 and 3.4

$$\|\nabla(z - \mathcal{I}_h z)\|_{L^2(\Omega)} \le c\left\|\sigma\nabla^2 z\right\|_{L^2(\Omega)} \le c\,|\ln h|\left\|\sigma\frac{\sigma^{-2}e_g}{\|\sigma^{-1}e_g\|}\right\|_{L^2(\Omega)} \le c\,|\ln h|\,. \tag{3.15}$$

Inequality (3.13) yields together with estimates (3.14) and (3.15) the assertion of this lemma. □

Lemma 3.7. *For the regularized Green function g^h and its Ritz projection g_h^h defined in* (3.11) *and* (3.12)*, respectively, the inequality*

$$\left\|\nabla^2(\sigma(g^h - g_h^h))\right\|_h \leq c\,|\ln h|^{3/2}$$

is satisfied.

Proof. We use again the abbreviation $e_g := g^h - g_h^h$, apply the product rule on every element $T \in \mathcal{T}_h$ and get

$$\nabla^2(\sigma e_g)|_T = (\nabla^2\sigma)e_g|_T + 2(\nabla\sigma)(\nabla e_g)|_T + \sigma(\nabla^2 e_g)|_T.$$

This results with Lemma 3.1 in the estimate

$$\left\|\nabla^2(\sigma e_g)\right\|_h^2 \leq c\left(\left\|\sigma^{-1}e_g\right\|_{L^2(\Omega)}^2 + \|(\nabla\sigma)(\nabla e_g)\|_{L^2(\Omega)}^2 + \left\|\sigma(\nabla^2 e_g)\right\|_h^2\right). \tag{3.16}$$

The first term of the right-hand side of this inequality is estimated in Lemma 3.6. For the second term one can conclude with the help of Lemma 3.1, Lemma 3.3 with the choice $v = g^h$ as well as with Lemma 3.4

$$\begin{aligned}
\|(\nabla\sigma)(\nabla e_g)\|_{L^2(\Omega)}^2 &\leq c\,\|\nabla e_g\|_{L^2(\Omega)}^2 \\
&\leq c\left\|\nabla(g^h - \mathcal{I}_h g^h)\right\|_{L^2(\Omega)}^2 \\
&\leq c\left\|\sigma\nabla^2 g^h\right\|_{L^2(\Omega)}^2 \\
&\leq c\,|\ln h|\,.
\end{aligned} \tag{3.17}$$

Since the equality $\nabla^2(g_h^h|_T) = 0$ holds on every element T for linear elements it follows for the third term with application of Lemma 3.5

$$\left\|\sigma(\nabla^2 e_g)\right\|_h^2 = \left\|\sigma\nabla^2 g^h\right\|_{L^2(\Omega)}^2 \leq c\,|\ln h|\,. \tag{3.18}$$

This means, Lemma 3.6 yields together with the inequalities (3.16), (3.17) and (3.18) the assertion. □

Lemma 3.8. *For the regularized Green function g^h and its Ritz projection g_h^h defined in* (3.11) *and* (3.12) *the inequality*

$$\left\|\sigma^{1/2}\nabla(g^h - g_h^h)\right\|_{L^2(\Omega)}^2 \leq ch\,|\ln h|^{3/2}$$

holds.

Proof. We use the abbreviation $e_g := g^h - g_h^h$. With the product rule we observe

$$\left\|\sigma^{1/2}\nabla e_g\right\|_{L^2(\Omega)}^2 = (\nabla e_g, \sigma\nabla e_g) = (\nabla e_g, \nabla(\sigma e_g)) - (\nabla e_g, e_g\nabla\sigma).$$

By introducing the nodal interpolant of σe_g,

$$\Psi_h := \mathcal{I}_h(\sigma e_g),$$

and applying Galerkin orthogonality one obtains that

$$\left\|\sigma^{1/2}\nabla e_g\right\|_{L^2(\Omega)}^2 =(\nabla e_g, \nabla(\sigma e_g - \Psi_h)) - (\nabla e_g, e_g \nabla\sigma). \tag{3.19}$$

For the first term of the right hand side of the last equation we estimate

$$\begin{aligned}(\nabla e_g, \nabla(\sigma e_g - \Psi_h)) &= (\sigma^{1/2}\nabla e_g, \sigma^{-1/2}\nabla(\sigma e_g - \Psi_h)) \\ &\leq \frac{1}{4}\left\|\sigma^{1/2}\nabla e_g\right\|_{L^2(\Omega)}^2 + \left\|\sigma^{-1/2}\nabla(\sigma e_g - \Psi_h)\right\|_{L^2(\Omega)}^2 \\ &\leq \frac{1}{4}\left\|\sigma^{1/2}\nabla e_g\right\|_{L^2(\Omega)}^2 + ch^2\left\|\nabla^2(\sigma e_g)\right\|_h^2,\end{aligned}$$

where we have used Lemma 3.2 in the last step. With Lemma 3.7 it follows

$$(\nabla e_g, \nabla(\sigma e_g - \Psi_h)) \leq \frac{1}{4}\left\|\sigma^{1/2}\nabla e_g\right\|_{L^2(\Omega)}^2 + ch^2\left|\ln h\right|^3. \tag{3.20}$$

For estimating the second term of the right hand side of (3.19) we consider another auxiliary equation,

$$-\Delta y = \frac{e_g}{\|e_g\|_{L^2(\Omega)}} \quad \text{in } \Omega, \qquad y = 0 \quad \text{on } \partial\Omega.$$

Utilizing the weak form of this equation with e_g as the test function, and later on Lemma 3.2, we can write

$$\begin{aligned}\|e_g\|_{L^2(\Omega)} &= (\nabla e_g, \nabla y) = (\nabla e, \nabla(y - \mathcal{I}_h y)) \\ &\leq \left\|\sigma^{1/2}\nabla e_g\right\|\left\|\sigma^{-1/2}\nabla(y - \mathcal{I}_h y)\right\| \\ &\leq \left\|\sigma^{1/2}\nabla e_g\right\| ch\left\|\nabla^2 y\right\| \\ &\leq ch\left\|\sigma^{1/2}\nabla e_g\right\|\end{aligned} \tag{3.21}$$

since the L^2-norm of $e_g/\|e_g\|_{L^2(\Omega)}$ is one. With this result the second term of the right-hand side of (3.19) can be estimated with the help of Lemma 3.1 as

$$\begin{aligned}(\nabla e_g, e_g\nabla\sigma) &= (\sigma^{1/2}\nabla e_g, \sigma^{-1/2}e_g\nabla\sigma) \\ &\leq \left\|\sigma^{1/2}\nabla e_g\right\|_{L^2(\Omega)}\left\|\sigma^{-1/2}e_g\nabla\sigma\right\|_{L^2(\Omega)} \\ &\leq \frac{1}{8}\left\|\sigma^{1/2}\nabla e_g\right\|_{L^2(\Omega)}^2 + c\left\|\sigma^{-1/2}e_g\right\|_{L^2(\Omega)}^2 \\ &\leq \frac{1}{8}\left\|\sigma^{1/2}\nabla e_g\right\|_{L^2(\Omega)}^2 + c(e_g, \sigma^{-1}e_g) \\ &\leq \frac{1}{8}\left\|\sigma^{1/2}\nabla e_g\right\|_{L^2(\Omega)}^2 + c\|e_g\|_{L^2(\Omega)}\left\|\sigma^{-1}e_g\right\|_{L^2(\Omega)}.\end{aligned}$$

With estimate (3.21) and Lemma 3.6 one can conclude

$$
\begin{aligned}
(\nabla e_g, e_g \nabla\sigma) &\leq \frac{1}{8}\left\|\sigma^{1/2}\nabla e_g\right\|_{L^2(\Omega)} + ch\left|\ln h\right|^{3/2}\left\|\sigma^{1/2}\nabla e_g\right\| \\
&\leq \frac{1}{4}\left\|\sigma^{1/2}\nabla e_g\right\|^2_{L^2(\Omega)} + ch^2\left|\ln h\right|^3
\end{aligned}
\tag{3.22}
$$

by applying Young's inequality in the last step. With equation (3.19) the assertion follows from inequalities (3.20) and (3.22). □

Now we are able to prove Theorem 2.1.

Proof. Let T^* denote an element that contains x_0, and set $\tilde{e} := z - z_h$. By using the nodal interpolant $\mathcal{I}_h$ we estimate

$$
\begin{aligned}
|(z - z_h)(x_0)| &\leq \max_{T^*}|\tilde{e}| \\
&\leq \max_{T^*}|z - \mathcal{I}_h z| + \max_{T^*}|\mathcal{I}_h\tilde{e}| \\
&\leq \max_{T^*}|z - \mathcal{I}_h z| + c\left|T^*\right|^{-1}\int_{T^*}|\mathcal{I}_h\tilde{e}| \\
&\leq \max_{T^*}|z - \mathcal{I}_h z| + c\left|T^*\right|^{-1}\left(\int_{T^*}|z - \mathcal{I}_h z| + \int_{T^*}|\tilde{e}|\right) \\
&\leq c\max_{T^*}|z - \mathcal{I}_h z| + c\left|T^*\right|^{-1}\int_{T^*}|\tilde{e}| \\
&\leq ch_*\left\|\nabla^2 z\right\|_{L^2(T^*)} + c\left|T^*\right|^{-1}\int_{T^*}|\tilde{e}|\,.
\end{aligned}
\tag{3.23}
$$

Since $h_* \sim h^2$ the first term obeys the claim of the theorem and it remains to estimate $|T^*|^{-1}\int_{T^*}|\tilde{e}|$. To this end, we consider the auxiliary problem (3.11). From the weak form of this equation is easy to see that

$$
(\nabla g^h, \nabla\tilde{e}) = (\delta^h, \tilde{e}) = \left|T^*\right|^{-1}\int_{T^*}|\tilde{e}| \tag{3.24}
$$

is the term left to consider. With the Ritz projection g_h^h defined in (3.12) we can write

$$
\begin{aligned}
(\nabla g^h, \nabla\tilde{e}) &= (\nabla(z - z_h), \nabla g^h) \\
&= (\nabla(z - z_h), \nabla(g^h - g_h^h)) \\
&= (\nabla(z - \mathcal{I}_h z), \nabla(g^h - g_h^h)) \\
&\leq \left\|\sigma^{-1/2}\nabla(z - \mathcal{I}_h z)\right\|_{L^2(\Omega)}\left\|\sigma^{1/2}\nabla(g^h - g_h^h)\right\|_{L^2(\Omega)},
\end{aligned}
\tag{3.25}
$$

using Galerkin orthogonality. The application of Lemmas 3.2 and 3.8 yields together with equation (3.24) the assertion. □

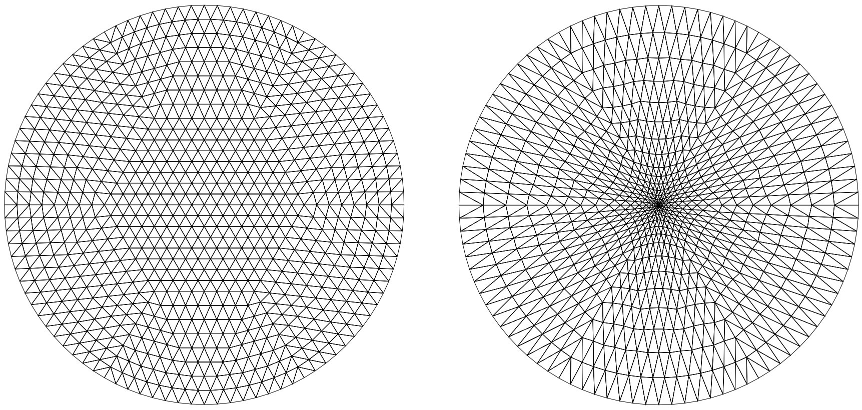

Figure 1: Quasi-uniform mesh and graded mesh according to (2.1)

4 Numerical example

In this section we illustrate our theoretical findings by a numerical example. We consider the boundary value problem (1.1) with

$$\Omega := \{x = (x_1, x_2) \in \mathbb{R}^2 \, : \, x_1^2 + x_2^2 < 1\}$$

and its finite element solution according to (2.2). We choose $x_0 = (0, 0)$, i.e. the right-hand side is the Dirac measure concentrated at the origin. Then the exact solution u is given as the fundamental solution of the Laplace equation,

$$u(x) = -\frac{1}{2\pi} \ln \sqrt{x_1^2 + x_2^2}.$$

For the computation of the finite element approximations we used the finite element library MooNMD [10]. We considered quasi-uniform meshes and meshes that are graded according to condition (2.1). In Figure 1 one can see both versions of meshes for $h = 1/16$. The graded mesh is generated by transforming the uniform mesh using the mapping

$$T(x) = x \, \|x\|^{-1/2} \, .$$

Table 1 shows the estimated order of convergence (eoc) for quasi-uniform and graded meshes, respectively. In the case of quasi-uniform meshes one can see a convergence rate of 1 in h as proved in [14]. For meshes that are graded according to condition (2.1) one observes a convergence rate slightly smaller than 2. This confirms our theoretical results. Notice that the curved boundary is approximated by straight lines. As the test has shown the additional error introduced by this has no influence on our results.

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Table 1: L^2-error and estimated order of convergence for quasi-uniform and graded meshes

	quasi-uniform mesh		graded mesh	
N_{nodes}	$\|u-u_h\|_{L^2(\Omega)}$	eoc	$\|u-u_h\|_{L^2(\Omega)}$	eoc
19	3.12e−02		3.88e−02	
61	1.48e−02	1.28	1.39e−02	1.76
217	7.29e−03	1.12	4.07e−03	1.93
817	3.63e−03	1.05	1.13e−03	1.93
3169	1.81e−03	1.02	3.06e−04	1.93
12481	9.06e−04	1.01	8.17e−05	1.93
49537	4.53e−04	1.01	2.16e−05	1.93
197377	2.27e−04	1.00	5.69e−06	1.93

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