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Treatment of boundary layers with anisotropic finite elements

The aim is to construct a finite element method which resolves layers appearing in singularly perturbed problems, here the easiest case of a reaction diffusion problem. For this, a special anisotropic mesh of Shishkin type is investigated. Estimates of the finite element error in the energy norm are derived which are uniformly valid with respect to the (small) diffusion parameter. A key ingredient for the error analysis is a refined estimate for derivatives of the interpolation error.

1. The model problem

Consider the reaction diffusion problem

$$-\varepsilon^2 \Delta u + cu = f \text{ in } \Omega = (0,1)^2, \quad u = 0 \text{ on } \partial\Omega, \tag{1}$$

where $\varepsilon \in (0,1]$ is the diffusion parameter, and c and f are sufficiently smooth functions, $c \geq c_0 > 0$. In the singularly perturbed case $\varepsilon \ll 1$ the solution of (1) is characterized by a boundary layer of width $\mathcal{O}(\varepsilon |\ln \varepsilon|)$. For the analysis of the finite element method we need localized Sobolev norm estimates of the solution with respect to ε. Unfortunately, such estimates are hard to obtain. The results of Shishkin [4] for smooth domains and for the unit square led us to an assumption which we are going to describe next.

Introduce a non-overlapping domain decomposition of Ω by introducing lines with a distance $b := b_0\varepsilon |\ln \varepsilon|$, $b_0 > \frac{3}{2c_0}$, to the boundary. Denote by $\Omega_1 := (b, 1-b)^2$ the interior subdomain, by $\Omega_2 := (0,b)^2 \cup (1-b,1) \times (0,b) \cup (1-b,1)^2 \cup (0,b) \times (1-b,b)$ the union of the small subdomains near the corners, and by $\Omega_3 := (b,1-b) \times (0,b) \cup (1-b,1) \times (b,1-b) \cup (b,1-b) \times (1-b,1) \cup (0,b) \times (b,1-b)$ the union of the remaining boundary strips. In Ω_3 we introduce a boundary fitted Cartesian coordinate system (x_1, x_2) with $x_2 := \text{dist}(x, \partial\Omega)$; derivatives D^α are to be understood with respect to this coordinate system.

We assume that the following estimates hold:

$$
\begin{align}
|u; W^{2,2}(\Omega_1)| &\leq C, \tag{2}\\
|u; W^{2,2}(\Omega_2)| &\leq Cb^{1/2}\varepsilon^{-3/2}, \tag{3}\\
\|D^\alpha u; L_2(\Omega_3)\| &\leq C(\varepsilon^{-1/2} + \varepsilon^{1/2-\alpha_2}), \quad |\alpha| = 2, \tag{4}\\
\|D^\alpha u; L_\infty(\Omega_3)\| &\leq C\varepsilon^{-\alpha_2}, \qquad\qquad |\alpha| = 1. \tag{5}
\end{align}
$$

A discussion of these assumptions can be found in [3, Subsection 2.2]. Here, we have used a multi-index notation with $\alpha = (\alpha_1, \alpha_2)$, $D^\alpha := \partial^{\alpha_1 + \alpha_2}/(\partial x_1^{\alpha_1} \partial x_2^{\alpha_2})$.

With $V := W_0^{1,2}(\Omega)$ the variational formulation of problem (1) reads:

$$\text{Find } u \in V \text{ such that } a(u,v) = (f,v) \text{ for all } v \in V, \tag{6}$$

where $a(u,v) := \varepsilon^2(\nabla u, \nabla v) + (cu, v)$ and $(.,.)$ is the $L_2(\Omega)$ inner product. Define by $\|| v \||_\Omega := \sqrt{a(v,v)}$ the energy norm of $v \in V$.

2. Finite element error estimates

Let $\mathcal{T}_h = \{e\}$ be a family of finite element meshes. Introduce the finite element space $V_h \subset V \cap C(\overline{\Omega})$ of all continuous functions which are linear in the triangular elements e. Then the finite element solution of (1) is defined by:

$$\text{Find } u_h \in V_h \text{ such that } a(u_h, v_h) = (f, v_h) \text{ for all } v_h \in V_h. \tag{7}$$

For motivating the use of anisotropic finite elements in the boundary strip Ω_3 we point out the following: If the domain was meshed using isotropic elements of equal size h, then the error estimate would be $\|| u - u_h \||_\Omega \leq Ch\varepsilon^{-1/2}$. If the boundary layer $\Omega_2 \cup \Omega_3$ was resolved using isotropic elements of diameter $\varepsilon^{1/2}h$ then one can obtain $\|| u - u_h \||_\Omega \leq Ch$. But then the number of elements grows to $\mathcal{O}(h^{-2}|\ln \varepsilon|)$.

As our favourite variant we propose to use in Ω_3 anisotropic elements with diameter $h_1 = h$ in tangential direction and $h_2 = bh$ normal to the boundary. We complete the mesh with isotropic elements of diameter h in Ω_1 and of diameter bh in Ω_2. Then we are able to prove

$$\| u - u_h \|_\Omega \leq Ch(\varepsilon^{1/2}|\ln\varepsilon| + h) \tag{8}$$

while using only $\mathcal{O}(h^{-2})$ elements. The proof can be sketched as follows:

In Ω_1 and Ω_2, we can use standard interpolation error estimates. Together with (2) and (3) we find

$$
\begin{aligned}
\| u - I_h u \|_{\Omega_1} &\leq C(\varepsilon h + h^2)|u; W^{2,2}(\Omega_1)| \\
&\leq C(\varepsilon h + h^2), \\
\| u - I_h u \|_{\Omega_2} &\leq C[\varepsilon bh + (bh)^2]|u; W^{2,2}(\Omega_1)| \\
&\leq C(\varepsilon bh + b^2 h^2)b^{1/2}\varepsilon^{-3/2} \\
&\leq Ch\varepsilon|\ln\varepsilon|^{3/2}(1 + h|\ln\varepsilon|).
\end{aligned}
$$

In Ω_3, we use the anisotropic interpolation error estimates

$$
\begin{aligned}
\|u - I_h u; L_2(e)\| &\leq C(\operatorname{mease})^{1/2}(h_1\|D^{(1,0)}u; L_\infty(e)\| + h_2\|D^{(0,1)}u; L_\infty(e)\|), \\
|u - I_h u, W^{1,2}(e)| &\leq C(h_1\|D^{(2,0)}u; L_2(e)\| + h_1\|D^{(1,1)}u; L_2(e)\| + h_2\|D^{(0,2)}u; L_2(e)\|).
\end{aligned}
$$

The second one is proved in [1], the first one can be obtained with the same methodology. We conclude with (4) and (5)

$$
\begin{aligned}
\| u - I_h u \|_{\Omega_3} &\leq C\varepsilon[h\varepsilon^{-1/2} + h\varepsilon^{-1/2} + (bh)\varepsilon^{-3/2}] + Cb[h + (ah)\varepsilon^{-1}] \\
&\leq Ch\varepsilon^{1/2}|\ln\varepsilon|.
\end{aligned}
$$

Summing up these estimates we obtain a global interpolation error estimate. By using the projection property of the finite element method we get the finite element error estimate (8). Note that the proof sketched here is slightly improved in comparison with [3]. The estimate (8) was derived there under an assumption which is slightly stronger than (4).

This approach can be generalized to arbitrary polygonal domains, to three dimensions, and to shape functions of higher order [3]. Quadrilateral elements can be used as well [2].

3. References

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