Anisotropic mesh refinement for the treatment of boundary layers*

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Abstract. For the approximation of anisotropic structures like edges and boundary/interior layers it is an obvious idea to use a family of finite element meshes with different mesh sizes in different directions, so called anisotropic meshes. The paper reviews anisotropic local interpolation error estimates for simplicial Lagrangian finite elements in two and three dimensions. Using these elements, an anisotropic mesh refinement strategy is justified for treating boundary layers in convection dominated problems.

Key Words. finite element method, anisotropic finite elements, interpolation error estimate, maximal angle condition, anisotropic mesh refinement.

AMS(MOS) subject classification. 65N30, 65N50.

1 Introduction

If the solution of a partial differential equation has different behaviour in different space directions then it is an obvious idea to reflect this in a finite element approximation by using a family of meshes with different mesh sizes in different directions, so-called anisotropic meshes. Applications include the approximation of edge and interface singularities in diffusion dominated problems (Poisson type equations) [2, 5], of boundary and interior layers arising in convection-dominated problems, see [4] and papers cited there, and of solutions of problems with strongly anisotropic material parameters.

In this paper we are concerned with the finite element solution of a linear(ized) diffusion-convection-reaction model in a bounded polyhedral domain $\Omega \subseteq I\!R^2$ with Lipschitz boundary $\partial\Omega$:

$$L_\varepsilon u \equiv -\varepsilon\Delta u + b\cdot\nabla u + cu = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega; \tag{1}$$

$\varepsilon \in (0,1]$ is a parameter. In case of $P(x) \equiv \varepsilon^{-1}\|b(x); R^d\| \gg 1$ and/or $\Gamma(x) \equiv \varepsilon^{-1}|c(x)| \gg 1$, problem (1) is of singularly perturbed type and the solution u may have sharp boundary or interior layers. The resolution of such layers is often the main interest in applications and will be considered here.

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Standard Galerkin finite element solutions may suffer from numerical instabilities which are generated by dominant convection and/or reaction terms unless the mesh is sufficiently refined. As a remedy, stabilized Galerkin methods have been proposed. We will focus here on the Galerkin/Least-squares method. We estimate the finite element error for a family of meshes with an a-priori anisotropic refinement in the layer regions, whereas an isotropic mesh away from the layers is used which could be (isotropically) refined via standard adaptive methods.

This family of meshes is *anisotropic* in the sense that $\lim_{\varepsilon\to+0} h_e/\varrho_e = \infty$, where h_e and ϱ_e denote the diameter of the finite element e and the diameter of the largest inscribed ball in e, respectively. But the usual condition $h_e/\varrho_e = \mathcal{O}(1)$ does not consider the anisotropic behaviour of the solution and leads to an overrefinement in the layers.

The numerical analysis of this method relies heavily on anisotropic interpolation error estimates. They are reviewed in Section 2. Using them we were able to prove that, for sufficiently smooth solutions u of (1), the finite element error $u-U_h$ converges in an energy type norm $|||\, v \,|||^2$ with the optimal order h^k almost uniform with respect to ε:

$$|||\, u - U_h \,||| \le C h^k |\ln \varepsilon|^{1/2}. \tag{2}$$

The meshes and the choice of the parameter set $\{\delta_e\}$ are descibed in Section 3.

2 Interpolation on anisotropic elements

Consider a polyhedral domain $\Omega \subset I\!R^d$, $d = 2, 3$, and let $\mathcal{T}_h = \{e\}$ be an admissible triangulation of $\overline{\Omega} = \bigcup_e \overline{e}$. Introduce the spaces V and V_h by

$$\begin{aligned} V &\equiv W_0^{1,2}(\Omega) \equiv \{v \in W^{1,2}(\Omega) : v|_{\partial\Omega} = 0\}, \\ V_h &\equiv \{v \in V : v|_e \in \mathcal{P}_k(e) \quad \forall e \in \mathcal{T}_h\}, \end{aligned}$$

where $\mathcal{P}_k$ is the space of polynomials of maximal degree $k \ge 1$. The interpolant $I_h^{(k)} v$ of a continuous function v is uniquely determined elementwise by $(I_h^{(k)} v)(x^{(i)}) = v(x^{(i)})$, $i = 1, \ldots, n$, $n = \dim \mathcal{P}_k$, where $x^{(i)}$ are the nodal points of the element e. Finally, let $W^{m,p}(e)$ $(m \in I\!N, p \in [1, \infty])$ be the usual Sobolev spaces with the norm and the special seminorm

$$\|v; W^{m,p}(e)\|^p \equiv \sum_{|\alpha|\le m} \int_e |D^\alpha v|^p dx, \qquad |v; W^{m,p}(e)|^p \equiv \sum_{|\alpha| = m} \int_e |D^\alpha v|^p dx,$$

and the usual modification for $p = \infty$.

Yet in the mid-seventies the question was addressed as to whether Zlámal's minimal angle condition can be weakened to a *maximal angle condition* [6, 7]. In two dimensions that means that the maximal interior angle γ_e of any element e is bounded by γ_*: $\gamma_e \le \gamma_*$, where $\gamma_* < \pi$ independent of h and $e \in \mathcal{T}_h$. But the interpolation results in these papers were rarely applied because the possible advantage of using elements with different diameters in different directions was not exploited. This remedy was removed by proving so-called anisotropic estimates.

To take advantage of the different sizes of the element e in different directions we introduce the following notation. For $e \subset I\!R^2$ let E_e be the longest edge of e. Then we denote by $h_{1,e} \equiv \mathrm{meas}_1(E_e)$ its length and by $h_{2,e} \equiv 2\,\mathrm{meas}_2(e)/h_{1,e}$ the diameter of e perpendicularly to E_e.

For anisotropic estimates we need a *coordinate system condition* which means in two dimensions that the angle ψ_e between E_e and the x_1-axis must be bounded in the sense $|\sin\psi_e| \le Ch_{2,e}/h_{1,e}$. In the three-dimensional case, the sizes $h_{1,e}$, $h_{2,e}$, and $h_{3,e}$, the maximal angle condition and the coordinate system condition are introduced similarly, see [3].

Theorem 1 *Assume that the element e fulfills the maximal angle condition and the coordinate system condition. Then for $v \in W^{k+1,p}(e)$ and $m = 0, \dots, k$, the estimate*

$$|v - I_h^{(k)}v; W^{m,p}(e)|^p \le C \sum_{|\alpha|=k+1-m} h_{1,e}^{\alpha_1 p} \cdots h_{d,e}^{\alpha_d p}\, |D^\alpha v; W^{m,p}(e)|^p$$

holds, if $d = 2$ or $m < k$ or $p > 2$.

The proofs are given in [3], special cases were derived before in [9] ($m = 0$) and [2] ($m = 1$). Note that (i) the restriction $p > 2$ is necessary in the case $d = 3$, $m = k$, (ii) the maximal angle condition is necessary for anisotropic interpolation, see for example [3, Section 7], (iii) refined estimates as given in Theorem 3 can also be proven for quadrilateral elements [1, 2], and for functions from weighted Sobolev spaces [5]. Furthermore, a numerical example that underlines the necessity of the coordinate system condition is given in [3, Section 7].

3 The convection diffusion reaction problem

We consider problem (1) with the basic assumptions $0 < \varepsilon \le 1$, $b \in [W^{1,\infty}(\Omega)]^d$, $f \in L^2(\Omega)$, and $\nabla\cdot b = 0$ almost everywhere in Ω. On the family of meshes $\mathcal{T}_h$ described below we introduce the following stabilized finite element method of Galerkin/Least-squares type:

$$\textit{Find } U_h \in V_h, \textit{ such that } B_{SG}(U_h, v_h) = L_{SG}(v_h) \quad \forall v_h \in V_h.$$

with

$$\begin{aligned} B_{SG}(u,v) &\equiv \varepsilon(\nabla u, \nabla v)_\Omega + \tfrac{1}{2}\{(b\cdot\nabla u, v)_\Omega - (b\cdot\nabla v, u)_\Omega\} + (cu,v)_\Omega + \\ &\quad + \sum_e \delta_e\,(L_\varepsilon u, L_\varepsilon v)_e, \\ L_{SG}(v) &\equiv (f,v)_\Omega + \sum_e \delta_e\,(f, L_\varepsilon v)_e, \end{aligned}$$

and a set $\{\delta_e\}$ of non-negative numerical diffusion parameters. Here, $(.\,,.)_G$ denotes the inner product in $L^2(G)$, $G \subseteq \Omega$. In view of the difficulties to get a priori information on the solution u we restrict our consideration to a certain class of problems introduced in Nävert [8] which allows to localize the boundary layers $\mathcal{R}_\varepsilon$ of thickness

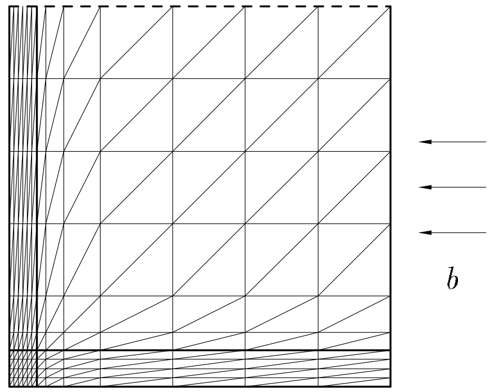

Figure 1: Anisotropic mesh in the boundary layer region

$\mathcal{O}(\varepsilon^\kappa \ln \frac{1}{\varepsilon})$, $\kappa = \frac{1}{2}$ or $\kappa = 1$, at some straight lines $M \subset \partial\Omega$. Then, we assume the following hypothesis is to be satisfied:

$$\|D^\alpha_{\eta,\xi} u; L^2(e)\| \leq K(f)\sqrt{\text{meas}(e)}\varepsilon^{-\kappa\alpha_2}.$$

We introduce now local coordinates (ξ, η) with $\xi = 0$ at M. As a starting point, we generate an orthogonal mesh via lines $\xi = \xi_i = i \cdot \varepsilon^\kappa h$, $\eta = \eta_j = j \cdot h$ $(i = 0, \ldots, i_0,$ $j = 0, \ldots, j_0)$ and $\xi_{i_0} = d(\varepsilon) \equiv \varepsilon^\kappa \ln \frac{1}{\varepsilon}$, see Figure 1. The rectangles $K = [\xi_i, \xi_{i+1}] \times [\eta_i, \eta_{i+1}]$ are split into 2 triangles which satisfy the maximal angle condition and the coordinate system condition with respect to the fitted coordinate system. Note that our approach guarantees a stronger refinement near corners of M. Outside $\mathcal{R}_\varepsilon$ we can double $h_{\xi,e}$ in ξ-direction (perpendicularly to $(\partial\Omega)_+$ and $(\partial\Omega)_0$, respectively) until $h_{\xi,e} \sim h$ but we can also omit this transition layer; it does not affect our analysis. We see easily that the number of elements is of the order $h^{-2}|\ln\varepsilon|^{-1}$.

Then we were able to prove the following error estimate in an energy type norm $|||\, v \,||| \equiv B_{SG}(v, v)$ [4].

Theorem 2 *Under the assumptions made above, and for $u \in H^{r+1,2}(\Omega)$, $1 \leq r \leq k$, and using the parameter design*

$$\delta_e = \frac{h^2_{\xi,e}}{\varepsilon\sqrt{1 + (P^{\text{an}}_e)^2 + (\Gamma^{\text{an}}_e)^2}} \quad \textit{if } (P^{\text{an}}_e)^2 \geq (\tilde{P}^{\text{an}}_e)^2 \equiv \sqrt{1 + (P^{\text{an}}_e)^2 + (\Gamma^{\text{an}}_e)^2},$$

$$\delta_e = \min\left\{\frac{\varepsilon}{\|b; [L^\infty(e)]^2\|};\ \frac{h^2_{\xi,e}}{\varepsilon} \cdot \frac{1 + (P^{\text{an}}_e)^2 + \Gamma^{\text{an}}_e}{1 + (P^{\text{an}}_e)^2 + (\Gamma^{\text{an}}_e)^2}\right\} \quad \textit{if } 0 \leq P^{\text{an}}_e \leq \tilde{P}^{\text{an}}_e,$$

with

$$P^{\text{an}}_e \equiv \frac{h_{\xi,e}\|b; [L^\infty(e)]^d\|}{\varepsilon}, \quad \Gamma^{\text{an}}_e \equiv \frac{h^2_{\xi,e}\|c; L^\infty(e)\|}{\varepsilon},$$

we get the almost uniform (with respect to ε) error estimate (2).

In a numerical example we found a resolution of a characteristic boundary layer with about 48 000 anisotropic elements of the same quality as with an isotropic uniform mesh with about 2 million elements.

An interior layer with known location can be treated in an analogous way, see [10] for an algorithm. If the position of an interior layer at some manifold M is not a-priori known, one can try to identify M in an adaptive procedure using an appropriate error indicator. Such an approach has been proposed in [11] for compressible flow problems. A (fixed) anisotropically refined mesh is then constructed in the neighbourhood of M.

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