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Incorporation of a-priori mesh grading into a-posteriori adaptive mesh refinement

Th. Apel* R. Mücke† J. R. Whiteman‡

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Abstract

In this paper we discuss a combined a-priori a-posteriori approach to mesh refinement in finite element methods for two- and three-dimensional elliptic boundary value problems containing boundary singularities. We review first both techniques of a-priori mesh grading around singularities and a-posteriori mesh refinement controlled by local error indicators. In examples of two- and three-dimensional boundary value problems we demonstrate the applicability and efficiency of various combined mesh refinement strategies.

1 Introduction

The quality of a finite element approximation to the solution of an elliptic boundary value problem can vary markedly over the computational domain. This is particularly the case when boundary singularities, arising from re-entrant corners and edges or from the change of the type of boundary conditions, are present. The deterioration of the approximation arises on account of the lower global regularity of the solutions in these situations as compared with problems having smooth boundaries and only one type of boundary condition. Many special numerical techniques have been developed in recent years to compensate for the effects of these *singularities*, and there is an extensive literature in this field, see e. g. [1, 2, 6, 11, 13, 14, 23, 27, 29, 30]. In this paper we shall focus on strategies, which are a combination of *a-priori* grading and *a-posteriori* (or adapted) mesh refinement techniques.

The *a-priori* local mesh grading approach has been analyzed mainly in the two-dimensional case [6, 24, 27, 29], but there are also some studies of three-dimensional contexts, see [1, 2, 4, 15, 16]. Based on analytical knowledge of the solution of the boundary value problem a family of meshes can be described which will produce optimal *a-priori* error estimates. The only information necessary for this is a lower estimate for the exponent β_1 in the singular part of the solution, for β_1 see (2.7). This technique can be applied with any finite element code. The only modification necessary is in the preprocessor to generate the *a-priori* graded mesh. It can be shown that the number of degrees of freedom for such a mesh is asymptotically the same as for ungraded meshes and that the asymptotic behaviour of the condition number of the resulting finite element stiffness matrix is not worse than that for problems with regular solutions [2, 4, 24, 27]. The disadvantage of the *a-priori* analysis is that it considers only the *asymptotic behaviour* of the finite element solution as the number of degrees of freedom tends to infinity. Nevertheless it is an important part of finite element analysis because it demonstrates the mesh which in this sense is optimal.

*Technische Universität Chemnitz-Zwickau, Fakultät für Mathematik, D–09107 Chemnitz, Germany; email: apel@mathematik.tu-chemnitz.de

†ABB Corporate Research, CH–5405 Baden-Dattwil, Switzerland; email: roland.muecke@chcrc.abb.ch

‡Brunel University of West London, BICOM, Institute of Computational Mathematics, Uxbridge, Middlesex, UB8 3PH, United Kingdom; email: john.whiteman@brunel.ac.uk

However, for detailed knowledge of the errors in a particular finite element approximation and for assessing its acceptability, an *a-posteriori* error estimate has to be provided. Since the first papers by *Babuška* and *Rheinboldt* [8, 9, 10] many different estimators have been developed and included in finite element codes, for a review and comparison see for example [22, 31, 32]. Usually *a-posteriori* error estimates are calculated locally and then amalgamated to form a global error estimate. They can thus also serve as an indicator for regions with large or small errors, respectively, and can be used to determine where a mesh has to be refined or even where it can be coarsened. This feature has brought a new dimension to finite element analysis, namely the creation of automatic mesh adapting finite element strategies. The process can be described as follows: Starting with a coarse initial mesh, the three steps

- calculating an approximate solution,
- estimating the error locally,
- generating an improved mesh,

are executed repeatedly until the estimated error is globally within a desired tolerance, for example 5% or 10% in engineering applications.

In the *h*-version of adaptivity, which we consider here, there are two main strategies for improving the mesh. The first is based on a subdivision of the existing elements. This is relatively easy to program, but has the disadvantage, that adjacent elements have only a small number of possible ratios of their mesh sizes, mainly 1 : 1 or 1 : 2. The second approach demands a complete remeshing on the basis of a mesh density function derived from the error estimator [25, 26] and it is necessary to have an automatic mesh generator working with this background information. In this case the meshes produced have a more gradually changing mesh size.

Especially in the first strategy, even though the sequence of meshes depends strongly on the initial mesh, often little attention is payed to an appropriate design of this mesh. In most cases *a-priori* knowledge of where the errors are large is totally ignored and not exploited in the design of the initial mesh.

The initial question of our investigation is whether savings in computational effort can be achieved by using mesh grading techniques combined with adaptive techniques. As a measure we shall use the number of refinement steps and the number of degrees of freedom required to achieve a finite element solution with an error below a given tolerance.

The outline of the paper is as follows: In the first two sections we state the class of problems to be considered and describe basic features of its finite element discretization. Particularly, in Subsection 3.2 we introduce the idea of appropriate mesh design for approximating functions of r^β-type, and in Subsection 3.3 we derive an error estimator by using the residuals of the finite element solution. In Sections 4 and 5 the combined application of *a-priori* mesh grading and *a-posteriori* adaptivity is demonstrated in the context of two- and three-dimensional problems, and finally conclusions are given. We should mention that the report [3] is an extended version of the paper in hand.

2 The model problem

2.1 Classical and weak formulation

We consider the quasi-harmonic differential equation

$$\Delta^* u + f \equiv \nabla \cdot (\mathbf{K} \cdot \nabla u) + f = 0 \quad \text{in} \quad \Omega \subset \mathrm{R}^{\mathrm{d}},\ d = 2, 3 \tag{2.1}$$

together with essential and natural boundary conditions

$$u = \overline{u} \quad \text{on} \quad \Gamma_u, \tag{2.2}$$

$$-\nabla^* u - \sigma u \equiv -(\mathbf{K} \cdot \nabla u) \cdot \mathbf{n} - \sigma u = \overline{q} \quad \text{on} \quad \Gamma_q, \tag{2.3}$$

where $\partial\Omega \equiv \Gamma = \Gamma_u \cup \Gamma_q$ and $\Gamma_u \cap \Gamma_q = \emptyset$. Boundary value problems like this occur in a wide range of physical processes, see for example [35]. Regarding the case of steady state heat conduction u denotes the temperature, and f the heat source in the body under investigation. On the boundary either the temperature is fixed or a defined heat outflow $\overline{q}$ is given. $\mathbf{K}$ is a symmetric and positive definite tensor of second order whose components K_{ij} describe the thermal conductivity of the physical medium, σ is the thermal conductance and $\mathbf{n}$ denotes the outward normal unit vector.

To obtain a weaker formulation of the boundary value problem which is appropriate for finite element techniques employed here we multiply the equilibrium conditions (2.1) and (2.3) by an arbitrary test function $v \in V$ where $V = \{v \in H^1(\Omega), v = 0 \text{ on } \Gamma_u\}$. After integration over Ω and Γ_q and employing *Green*'s theorem of integration by parts we obtain the weak formulation of the boundary value problem as follows: Find an $u \in \overline{V}$ such that

$$a(u,v) = b(v) \qquad \forall\, v \in V\,. \tag{2.4}$$

Here, $\overline{V} = \{u \in H^1(\Omega), u = \overline{u} \text{ on } \Gamma_u\}$ and

$$a(u,v) = \int_\Omega \nabla u \cdot \mathbf{K} \cdot \nabla v \, d\Omega + \int_{\Gamma_q} \sigma\, u\, v \, d\Gamma\,, \tag{2.5}$$

$$b(v) = \int_\Omega f\, v \, d\Omega - \int_{\Gamma_q} \overline{q}\, v \, d\Gamma\,, \tag{2.6}$$

are bilinear and linear forms defined on $H^1 \times H^1$ and H^1, respectively. Notice, that $\Gamma_u \neq \emptyset$ or $\sigma \neq 0$ is required for the existence of a unique solution u. The energy norm on V is defined by $\|.\|_E \equiv \sqrt{a(.,.)}$.

2.2 Singular solutions

The regularity of the solution of problem (2.4) is determined by the smoothness of the coefficients K_{ij} and the right-hand sides f, $\overline{u}$, and $\overline{q}$, as well as by the properties of the domain. For sufficiently smooth domains and coefficients the so-called shift theorem holds; that means, that for $k \geq 0$

$$u \in H^{k+1}(\Omega) \quad \text{if} \quad f \in H^{k-1}(\Omega),\ \overline{q} \in H^{k-1/2}(\Gamma_q),\ \overline{u} \in H^{k+1/2}(\Gamma_u)\,.$$

This is no longer true, when the domain Ω contains corners or edges. Consider for simplicity smooth data, we let K_{ij}, $f \in C^\infty(\overline{\Omega})$, and σ, $\overline{q}$, $\overline{u}$ be traces of $C^\infty(\overline{\Omega})$-functions with respect to Γ_q and Γ_u, respectively. Then in the case of a two-dimensional domain the solution u can in general be represented by

$$u = \xi(r) \sum_i \gamma_i r^{\beta_i} \Phi_i(\varphi) + u_r\,. \tag{2.7}$$

Here, r and φ are polar co-ordinates centered at the singularity, the constants γ_i denote *stress intensity factors*, $\xi(.)$ is a cut-off function, $\Phi_i(.)$ are trigonometric functions, $\beta_i \in (0,1)$ are real numbers, and u_r is the regular part of the solution. The exponents β_i depend on the opening angle at the corner and the coefficients K_{ij}, and can be determined exactly. — Note that there are some exceptional angles which depend on the coefficients K_{ij}. In these cases the representation formula (2.7) is not valid, and additional logarithmic terms must be included.

In three dimensions, the irregular boundary points are classified as conical corners, edges and polyhedral corners. Near conical points the solution u behaves as in the two-dimensional case. The radius r is here the distance to the corner and the functions Φ_i depend now on two spherical co-ordinates, i. e.

$$u = \xi(r) \sum_i \gamma_i r^{\beta_i} \Phi_i(\varphi, \theta) + u_r\,. \tag{2.8}$$

For $\mathbf{K} = \mathbf{I}$ the exponents β_i can be determined exactly [13].

Near edges we have also a representation formula similar to (2.7). Here, r is the distance to the edge, but the coefficients γ_i are no longer constants. Denote by z the co-ordinate in the direction of the edge, then for constant coefficients K_{ij} and a constant interior angle of the edge

$$u = \xi(r) \sum_i \gamma_i(z) r^{\beta_i} \Phi_i(\varphi) + u_r \,. \tag{2.9}$$

In the case of polyhedral corners we have a superposition of corner and edge singularities. The additional difficulty is that the functions $\Phi_i(\varphi, \theta)$ of the spherical co-ordinates are no longer smooth. We remark that this situation gets still more complicated when the data is not smooth and more general edges are considered. These problems are excluded here. For a detailed mathematical investigation of boundary singularities we refer to the monographs [18, 19].

3 Finite element discretization

3.1 Asymptotic error and extrapolation techniques

In conforming finite element techniques considered here the set of C^0-continuous shape functions $\overline{V}^h = \text{span}\{N_i\}_{i=1}^n$ (n denotes the total number of nodes) is a subset of $\overline{V}$, i. e. $\overline{V}^h \subset \overline{V}$. Thereby the weak formulation (2.4) of the boundary value problem is projected into finite dimensions as follows: Find $u^h \in \overline{V}^h$ such that

$$a(u^h, v^h) = b(v^h) \qquad \forall\, v^h \in V^h \,. \tag{3.1}$$

The difference between the finite element solution u^h and the exact solution u is represented by the error function $e = u - u^h$. It is well known that in a sequence of sufficiently fine finite element meshes the global error in the energy norm has the asymptotic behaviour

$$\|e\|_E^2 = a(e,e) \approx C^* N^{-\alpha} \,. \tag{3.2}$$

Here, N is the number of degrees of freedom of the finite element discretization, and C^* denotes a constant which is independent of N but influenced by the solution domain, the regularity of the exact solution, the polynomial order of the finite element shape functions and the mesh geometry. The convergence order α of the finite element solution for a sequence of uniformly refined meshes is given by

$$\alpha = \frac{2}{d} \min(s - 1, p) \tag{3.3}$$

where d is the dimension of the problem considered, s is the highest order of the *Sobolev-Slobodetskiĭ* space which contains the exact solution and p denotes the polynomial order of shape functions [12]. Obviously, in problems with singularities in the gradient of the solution, i. e. $s < 2$, the convergence order is determined by the regularity s of the exact solution.

If we assume that $\overline{u} = 0$ and that the convergence behaviour is already in the asymptotic range so that (3.2) has become valid, we can use the identity

$$a(e,e) = a(u,u) - a(u^h, u^h) \,. \tag{3.4}$$

to extrapolate the unknown energy $a(u,u)$ as follows: Determine the convergence order α either *a-priori* by (3.3) (if the regularity s is known) or *a-posteriori* by error estimators η which should have asymptotically the same convergence order as the energy

$$\alpha \approx \frac{\ln \eta_1^2 - \ln \eta_2^2}{\ln N_2 - \ln N_1} \,. \tag{3.5}$$

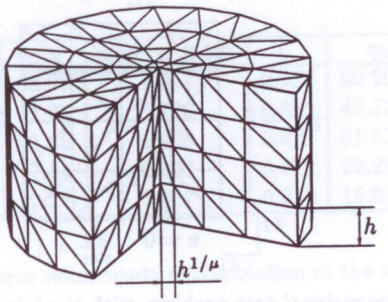


Figure 3.1: Anisotropic, graded mesh near an edge.

Using two finite element meshes with (3.2) and (3.4) the extrapolated energy $a_{ex}(u,u) \approx a(u,u)$ is then given by

$$a_{ex}(u,u) = \frac{a(u_1^h,u_1^h) - a(u_2^h,u_2^h)\left(\frac{N_2}{N_1}\right)^{\alpha}}{1-\left(\frac{N_2}{N_1}\right)^{\alpha}}. \tag{3.6}$$

3.2 A-priori mesh grading

Assume that the solution u of a two-dimensional domain can be represented as in (2.7). For simplicity we consider such solutions which have a regular part $u_r \in H^2(\Omega)$ and only one singular term. We now follow the idea of *Oganesyan* and *Rukhovets* [23] and employ the co-ordinate transformation $r^{\mu} = \varrho, \quad \mu \in (0,1]$. Thereby the singular part of the solution is transformed into

$$u_s = u_s(\varrho,\varphi) = \gamma\bar{\xi}(\varrho)\varrho^{\beta/\mu}\Phi(\varphi) \tag{3.7}$$

so that, in contrast to $\frac{\partial^2 u_s}{\partial r^2}$, the derivatives $\frac{\partial^k u_s}{\partial \varrho^k}$ $(k = 2,3,\ldots)$ are square-integrable for sufficiently small values of μ $(\mu < \frac{\beta}{k})$. We can suppose now that u_s can be approximated on a typical quasiuniform mesh with optimal order.

Trying to avoid this co-ordinate transformation for practical calculations we only transform the finite element mesh as follows: For all nodes within an appropriate radius b around the singularity centered at (x_{s_1}, x_{s_2}) the new nodal co-ordinates are calculated as

$$x_{i,\text{new}} = x_{i,\text{old}} \cdot \left(\frac{r}{b}\right)^{\frac{1}{\mu}-1} \qquad i = 1,2$$

where $r = [(x_1 - x_{s_1})^2 + (x_2 - x_{s_2})^2]^{1/2}$ (circular region) or $r = |x_1 - x_{s_1}| + |x_2 - x_{s_2}|$ (quadrilateral region). For a more detailed consideration see [2, 3, 4].

Estimates of the discretization error are derived from different points of view in several papers including [4, 11, 15, 16, 18, 24, 27, 29]. Whilst the proofs are given only for special cases, it can be conjectured that finite element discretizations with polynomial shape functions of degree p have the same approximation order as problems without singularities if the condition $\mu < \mu_* \equiv \beta/p$ is satisfied.

It is a natural idea to apply the mesh grading approach also to problems in three-dimensional domains, for example with edges. But then the following situation arises: the idea of transformation, mesh generation and re-transformation leads to anisotropic elements near the edges, see Figure 3.1. According to [1] an element is called *anisotropic* if its diameters in different directions have different asymptotics as the element size is tending to zero. Such meshes have already been applied successfully. For linear elements, and with certain smoothness assumptions on the data, it has been shown in [1] for three-dimensional domains that the optimal order of convergence is achieved for $\mu < \beta$.

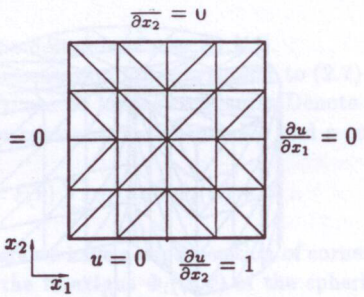


Figure 4.1: The two-dimensional test problem with the initial finite element mesh (domain: 100×100).

3.3 A-posteriori error estimation

Several techniques of *a-posteriori* error estimation have been elaborated and, based on the number of papers published on this field the development has been much pursued during the last few years. In this paper we employ a residual type error estimator. It is mainly connected with the works of *Babuška* and his co-workers, and uses the duality between the classical and the weak formulation of the boundary value problem [5, 7, 8, 9, 10, 20, 34]. The finite element solution, which satisfies the continuity requirements of the solution of the weak form (2.4) but not those of the solution of the classical formulation (2.1–2.3), produces residuals in the equilibrium conditions (2.1) and (2.3) and these are employed for assessing the discretization error of the finite element solution, for more details see also [3, 21].

Following this approach the discretization error in the energy norm is estimated by

$$\|e\|_E^2 \approx \eta^2 = \sum_{i=1}^{m} \lambda_i^2 = C^2 \sum_{i=1}^{m} \frac{1}{E(\mathbf{K}_i)} \sum_j h_{ij} \int_{(\partial\Omega_i)_j} \overline{J}^2 \, d\Gamma \tag{3.8}$$

with

$$\overline{J} = \begin{cases} 0 & \text{on } \overline{\Omega}_i \cap \Gamma_u \,, \\ \xi & \text{on } \overline{\Omega}_i \cap \Gamma_q \,, \\ \frac{1}{2} J & \text{on } \overline{\Omega}_i \cap \overline{\Omega}_j \,. \end{cases} \tag{3.9}$$

Here, the quantity η is the global error estimator whilst the indicator λ_i estimates the contribution of an individual element to the global error. The residual $\xi = -\nabla^* u^h - \sigma u^h - \overline{q}$ is defined along the surface part Γ_q and $J = (\mathbf{K}_i \cdot \nabla u^h) \cdot \mathbf{n}_i + (\mathbf{K}_j \cdot \nabla u^h) \cdot \mathbf{n}_j$ denotes the residual on the interface between two adjacent elements Ω_i and Ω_j. C is a global constant which we estimate by *Richardson* extrapolation and $E(\mathbf{K}_i)$ denotes the largest eigenvalue of the tensor $\mathbf{K}$ which can vary from element to element. $(\partial\Omega_i)_j$ stands for the interface between two adjacent elements Ω_i and Ω_j. In two-dimensional domains the discretization parameter h_{ij} is taken as $h_{ij} = \text{meas}(\partial\Omega_i)_j$. In three dimensions we use two versions:

$$h_{ij} = (2\,\text{meas}((\partial\Omega_i)_j))^{1/2}, \tag{3.10}$$

$$h_{ij} = (6\,\text{meas}(\Omega_i))^{1/3}, \tag{3.11}$$

related to the area of the faces and to the volume of the element, respectively.

4 Mesh grading and adaptivity in two-dimensional problems

We consider the two-dimensional *Laplace* equation, $\Delta u = 0$, together with boundary conditions illustrated in Figure 4.1. The change of the type of boundary condition on the lower surface of the domain causes a singularity in the gradient of the solution of order $O(r^{-\frac{1}{2}})$, see Subsection 2.2. The discretization error of the finite element solution based on the mesh

N	$\eta_{C=1}$	$a(u^h,u^h)/10^3$	C	$\eta_{\%}$
22	38.93	1.411	0.727	60.20 %
76	31.60	1.790	0.651	43.72 %
280	23.63	1.996	0.623	31.31 %
1072	17.14	2.103	0.612	22.29 %
4192	12.27	2.157	0.610	15.91 %

Table 4.1: Uniform refinement: extrapolation of the scale factor C.

shown in Figure 4.1 is expected to be significant near this singular point. In order to decrease this error as cheap as possible we treat the domain around the singularity by an appropriate mesh refinement. We discuss and compare here the following strategies:

- uniform mesh refinement,
- uniform mesh refinement with mesh grading,
- adaptive mesh refinement,
- adaptive mesh refinement with mesh grading.

In uniform mesh refinement each element is subdivided into four congruent elements. Obviously, this strategy is not very powerful because both the *a-priori* information about the exact solution around the singularity and the *a-posteriori* estimation of the discretization error distribution of the finite element solution are not included in this procedure. Mesh grading techniques as discussed in Subsection 3.2 make use of the known exponent of singularity to improve *a-priori* the mesh design. In contrast to this, adaptive mesh refinements are controlled *a-posteriori* by estimating the distribution of the discretization error of the finite element solution. Based on this only those elements are subdivided in which the estimated error is high.

To estimate the global constant C by extrapolation we consider first the behaviour of the error estimator η for a sequence of uniformly refined meshes. The numerical results we discuss are summarized in Table 4.1. According to (3.3) the theoretical convergence order of the discretization error becomes $\alpha = \frac{1}{2}$ for the problem under investigation. Using extrapolation (3.6) we determine the extrapolated energy as $a_{ex}(u,u) \approx 2.213 \cdot 10^3$. Now we can calculate the scale factor C by

$$C = \frac{\|e\|_E}{\eta_{C=1}} = \left(\frac{a_{ex}(u,u) - a(u^h,u^h)}{\eta^2_{C=1}} \right)^{\frac{1}{2}} . \tag{4.1}$$

The numerical results in Table 4.1 demonstrate that for sufficiently refined meshes a reasonable constant can be found.

In Figure 4.2 the behaviour of the discretization error using the refinement techniques mentioned above is demonstrated. Corresponding finite element meshes are shown in Figure 4.3. According to Section 3.2 an appropriate grading parameter employed here is given as $\mu = 0.4$.

With uniform mesh refinement the convergence order depends on the regularity of the exact solution. By refinement new degrees of freedom are created uniformly distributed over the domain so that neither the structure of the exact solution nor the distribution of the finite element discretization error is taken into consideration. Therefore with uniform mesh refinement the highest discretization error remains around the singularity. In Figure 4.3(a) the mesh after 3 uniform refinements (1072 degrees of freedom) is shown.

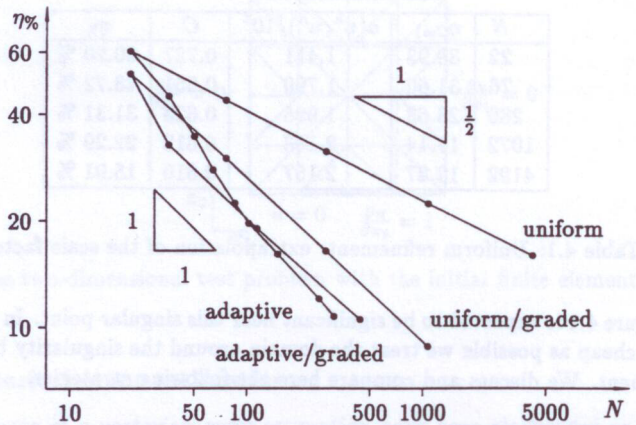
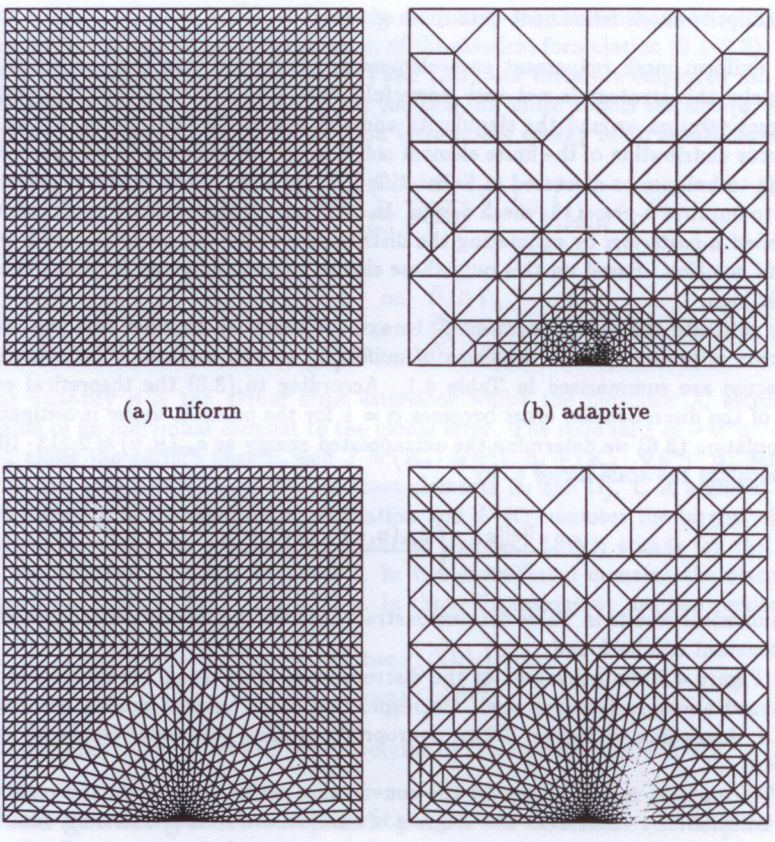


Figure 4.2: Behaviour of the discretization error by using various mesh refinement techniques.



(a) uniform (b) adaptive

(c) uniform with grading (d) adaptive with grading

Figure 4.3: Finite element meshes after various refinement strategies.

M	N	$\|e\|_E$	h_{ij} as in (3.10)		h_{ij} as in (3.11)	
			$\eta_{C=1}$	$\|e\|_E/\eta_{C=1}$	$\eta_{C=1}$	$\|e\|_E/\eta_{C=1}$
3	20	2.3391	10.5970	0.221	10.4270	0.224
6	275	1.4915	6.7680	0.220	6.6615	0.224
12	2783	0.9444	4.2717	0.221	4.2060	0.225
24	24863	0.5965	2.6912	0.222	2.6504	0.225

Table 5.1: Behaviour of the error estimator for $\mu = 1.0$.

In adaptive techniques the influence of the singularity on the convergence order is eliminated. Controlled by the error indicators a mesh refinement is produced especially near the singularity. We reach a global convergence order of approximately one, which is optimal for finite element approximations with linear shape functions. With classical adaptive techniques, however, the mesh refinement is *locally uniform*, i. e. an element which has to be refined is divided in four congruent subelements. The information about localisation and order of singularities to control *a-priori* a graded mesh design is not included. Therefore usually a large number of refinement steps is needed to achieve a sufficiently fine mesh around the singular point.

The uniform mesh refinement with mesh grading leads to the same optimal convergence order as in the case of adaptive refinement. By an appropriate mesh grading of the uniform refined mesh the influence of singularity on the convergence order of the finite element solution is eliminated. Notice, that already in the initial mesh a mesh grading reduces the discretization error, in this case from 60.20% to 51.93%, see Figure 4.2.

The most efficient mesh refinement technique is obviously the adaptive mesh refinement combined with mesh grading around the singularity. The achieved convergence order is near one which has already been obtained with uniform graded refinement as well as with classical adaptive refinement. In comparison with uniform graded refinement, however, the error level is now decreased and in contrast to classical adaptive refinement the number of meshes needed to achieve a certain error level is substantially reduced.

5 Numerical results in a three-dimensional domain

In the three-dimensional domain $\Omega = \{(x_1, x_2, x_3) = (r\cos\varphi, r\sin\varphi, z) \in \mathrm{R}^3 : r < 1, 0 < \varphi < \frac{3}{2}\pi, 0 < z < 1\}$ we consider *Laplace*'s equation $\Delta u = 0$ with essential boundary conditions $u = \overline{u}$ on $\partial\Omega$. The right hand side $\overline{u}$ is taken such that

$$u = (10 + z)r^{2/3}\sin\frac{2}{3}\varphi \tag{5.1}$$

is the exact solution of the problem which has the typical singular behaviour at the edge.

First we investigate the influence of mesh grading on the convergence behaviour of the finite element error. In Figure 5.1 it is illustrated that the error is decreasing with decreasing values of μ until some optimal μ_* of about 0.5, then it increases again. Note also that $\mu = 0.4$ and $\mu = 0.3$ give a relatively large error for coarse meshes. We can conclude here that the anisotropic, graded meshes are useful for treating edge singularities, for diminishing the error and achieving the optimal approximation order.

In Table 5.1 and Table 5.2 we set out, for meshes with different numbers M of layers, the exact error norm and the estimates $\eta_{C=1}$ in both variants of the discretization parameter h_{ij} in (3.8). For ungraded meshes ($\mu = 1.0$) the ratio $\|e\|_E/\eta_{C=1}$ is almost independent of the mesh size, so that extrapolation techniques as described in Subsection 3.1 will give a reasonable constant.

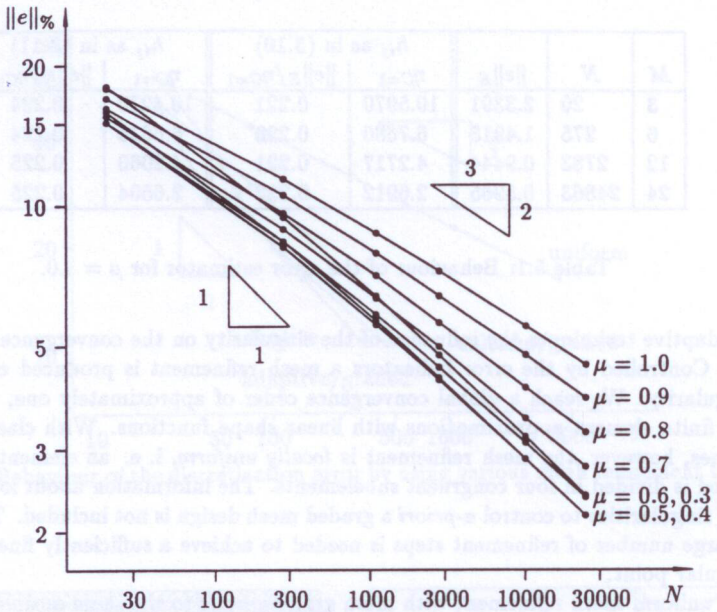


Figure 5.1: Behaviour of the error for different grading parameters μ.

			h_{ij} as in (3.10)		h_{ij} as in (3.11)	
M	N	$\|e\|_E$	$\eta_{C=1}$	$\|e\|_E/\eta_{C=1}$	$\eta_{C=1}$	$\|e\|_E/\eta_{C=1}$
3	20	1.9886	10.5930	0.188	9.9354	0.200
6	275	1.0655	6.1399	0.174	5.6382	0.189
12	2783	0.5562	3.4363	0.162	3.0910	0.180
24	24863	0.2858	1.8775	0.152	1.6555	0.173

Table 5.2: Behaviour of the error estimator for $\mu = 0.5$.

As illustrated in Table 5.2, in uniform graded meshes ($\mu < 1$) the ratio $\|e\|_E/\eta_{C=1}$ changes significantly with the number M of layers. We see the reason for this in the unbounded aspect ratio, that is the ratio between the radius of the smallest outer and the largest inner balls of the element. Note that the aspect ratio is of order $N^{3(-1+1/\mu)}$ for elements near the edge, compare Figure 3.1. — A modification of the error estimator is necessary, for example by employing another discretization parameter h_{ij} in (3.8). Because the exact solution of the boundary value problem under investigation is given in (5.1), we can also use the exact error function $e = u - u^h$ to indicate the local discretization error.

In further computational tests we have applied mesh grading and adaptivity together in the three-dimensional context. Therefore we have varied the grading parameter μ between 0.3 and 0.7, the desired error tolerance between 3% and 10%, as well as the mesh size of the initial mesh. Here we present the cases $\mu = 0.5$, $\varepsilon = 0.05\,\|u^h\|_E$ and initial meshes with 3 layers.

We have used two strategies of involving mesh grading, see Figures 5.2 and 5.3. In the first one only the initial mesh was graded. The new nodes introduced in the refinement process, are generated at the midpoints of the edges of the element of the previous level. The disadvantage of this procedure is that the effect of grading gets partially lost during the refinement process. The results illustrated in Figure 5.2 show exactly this behaviour.

In a second strategy the refined meshes were graded as well. This was realised by two

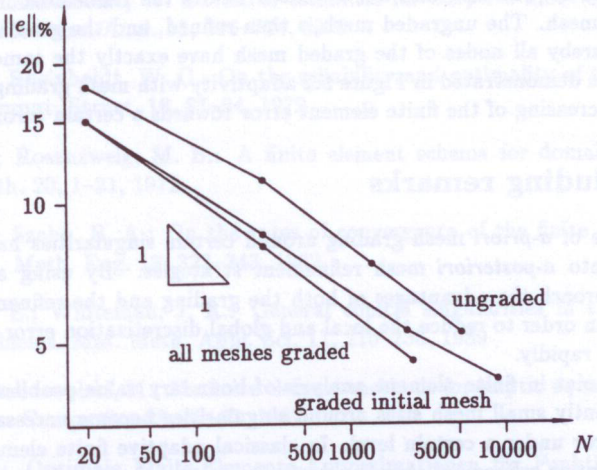
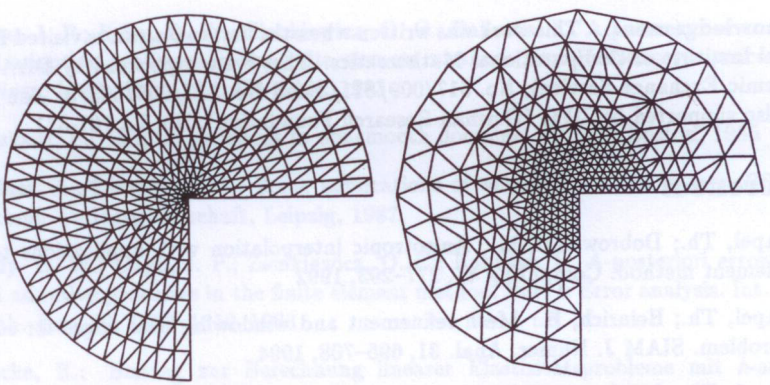
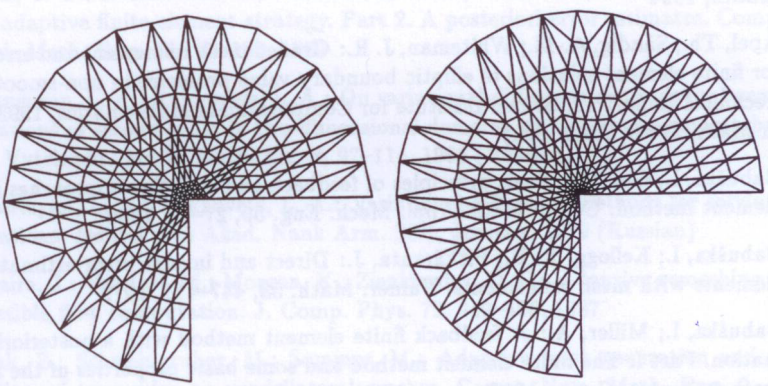


Figure 5.2: Error in the energy norm, initial mesh with 3 layers.



(a) uniform, graded, $M = 12$ (b) adaptive, without grading



(c) adaptive, graded initial mesh (d) adaptive, all meshes graded

Figure 5.3: Cross-cuts through final meshes at $z = \frac{1}{3}$.

node movements. The first one was executed before the refinement in order to reproduce an ungraded mesh. The ungraded mesh is then refined, and the grading is produced again. Note that thereby all nodes of the graded mesh have exactly the same co-ordinates in the next mesh. As demonstrated in Figure 5.2 adaptivity with mesh grading in each mesh shows the fastest decreasing of the finite element error towards a certain error level.

6 Concluding remarks

The technique of *a-priori* mesh grading around certain singularities has been incorporated successfully into *a-posteriori* mesh refinement strategies. By using a unified *a-priori a-posteriori* approach the advantages of both the grading and the refinement procedures can be combined in order to reduce the local and global discretization error of the finite element solution more rapidly.

The key point in finite element analysis of boundary value problems with singularities is that sufficiently small mesh sizes around singularities become necessary to bring the discretization error under a certain level. In classical adaptive finite element methods, where with each adaptive refinement step the mesh size around the singularity can be decreased at most by one half, usually a large number of steps is needed to achieve the desired error level. Adaptive refinement combined with mesh grading seems to be a suitable tool to improve finite element meshes around singularities faster than without this feature.

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